School of Astrophysics “Francesco Lucchin” Dynamical Evolution of Globular Clusters

Lecture 1: Theoretical Framework of Globular Cluster Dynamical Evolution

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1 Introduction

Globular clusters (see Fig. 1) have long been regarded as near-perfect laboratories for studies of stellar physics and stellar dynamics. Some reasons (and complications) are:

- they are isolated in space (but not all clusters are found in galactic halos—many disk and bulge clusters are known, and dynamical friction has probably transported many clusters to the Galactic Center)
- they contain coeval stars (but many clusters are now known to contain multiple stellar populations indicating several distinct phases of star formation)
- they contain virtually no gas or dust (today, that is—at early times, gas dynamical processes dominated their evolution)
- they are nearly spherical (although several are measurably flattened by rotation and/or tidal effects)

These systems thus represent a relatively—although not perfectly—“clean” realization of the classical N-body problem

$$\mathbf{a}_i \;\equiv\; \ddot{\mathbf{x}}_i \;=\; \sum_{j\neq i}^{N} Gm_j \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|^3}, \quad i = 1, \ldots, N. \tag{1}$$

We begin our study of cluster dynamics by ignoring complicating factors such as gas dynamics, stellar evolution, mass loss, etc., and focus on the pure N-body problem, basically as it might have been described by Newton. We'll define time scales and other units, discuss the fundamental dynamical processes driving cluster evolution, and present some basic terminology relevant to cluster dynamics.

Eq. (1) is a *particle* description of the problem—appropriate to star clusters with $N \sim 10^3 - 10^7$. For sufficiently large N (and $N = 10^6$ is probably enough), it is possible and convenient to switch to a continuum description of the problem, in which we treat the stars as a continuous fluid, with well defined “thermodynamic” properties $\rho(\mathbf{x})$, $\langle v^2\rangle(\mathbf{x})$, etc. The condition for this to be feasible, for N stars uniformly distributed in a box of side L, which represents the characteristic length scale of the cluster—the macroscopic scale of interest—is that there exist a scale $\lambda \ll L$ such that a cube of side λ contains many particles,

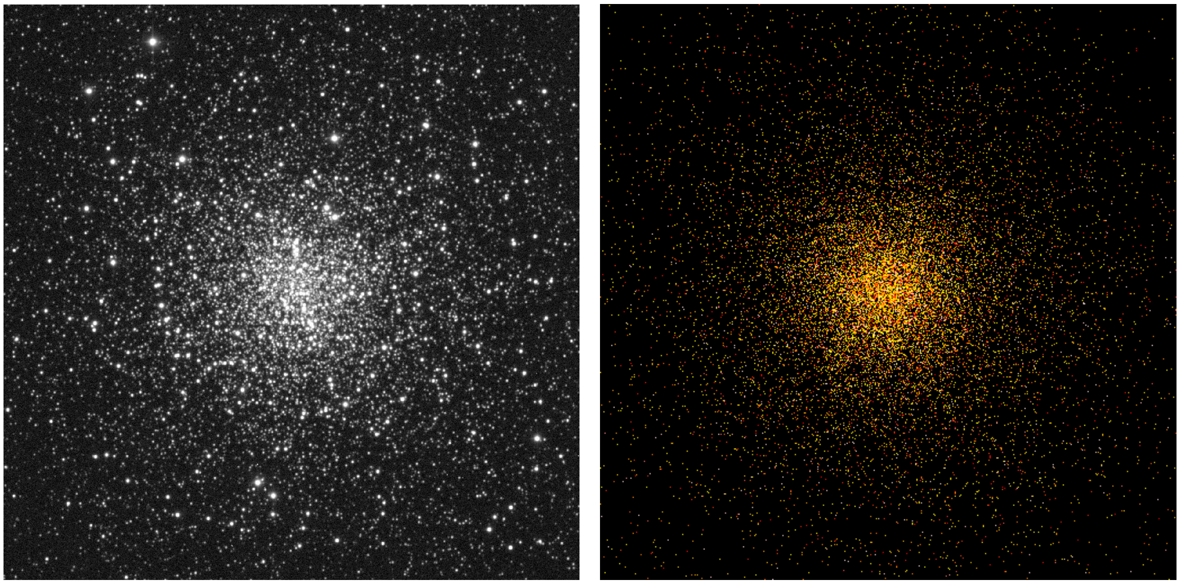

Figure 1: Left: an HST image of the old globular cluster M4. Right: an N-body realization (without red giants) of a cluster having similar structural properties.

so meaningful thermodynamic definitions are possible. In terms of the mean interparticle spacing $\ell \sim L/N^{1/3}$, we require that

$$\ell \ll \lambda \ll L. \tag{2}$$

If this is the case, we can employ the fluid approximation on scales larger than λ.

For now, we will adopt the more natural, and arguably more physical, particle view, but will briefly return to the continuum formulation later. One very interesting, and still current, aspect of cluster dynamics is the remarkable degree to which the continuum description captures the essential dynamics, even for quite small systems where the approximation is hard to justify. See §4 and Heggie's lectures.

2 Virial Equilibrium

Let's begin by refining our notion of "equilibrium" for a stellar system. For a fluid-dynamical system, equilibrium means the familiar hydrostatic equilibrium, in which the fluid is at rest (no large-scale velocities—on scales greater than λ, as just defined). In a stellar system, stars are constantly moving in and out on their orbits, but the analogous statement is that, at any given radius (for a spherical system, at least), there are as many stars moving inward as moving outward—that is, there is no net radial stellar flux.

2.1 The Virial Theorem

A convenient global restatement of the above condition involves the quantity

$$I = \sum_{i=1}^{N} m_i \, r_i^2,$$

where $r_i = |\mathbf{x}_i|$. This is the radial moment inertia of the system. Differentiating, we have

$$\begin{aligned} \dot{I} &= 2 \sum_{i=1}^{N} m_i \, \mathbf{x}_i \cdot \mathbf{v}_i \, , \\ \ddot{I} &= 2 \sum_{i=1}^{N} m_i \left(v_i^2 + \mathbf{x}_i \cdot \mathbf{a}_i \right) . \end{aligned}$$

Setting $\ddot{I} = 0$ as our definition of equilibrium, we have

$$\sum_{i=1}^{N} m_i \, v_i^2 + \sum_{i=1}^{N} m_i \, \mathbf{x}_i \cdot \mathbf{a}_i \;=\; 0.$$

The first term is simply twice the total kinetic energy of the system, $2\mathcal{T}$. We can simplify the second term by using Eq. (1) to rewrite it as

$$\begin{aligned} \sum_{i=1}^{N} \sum_{j \neq i}^{N} G m_i m_j \frac{\mathbf{x}_i \cdot (\mathbf{x}_j - \mathbf{x}_i)}{|\mathbf{x}_j - \mathbf{x}_i|^3} &= \sum_{i=1}^{N} \sum_{j > i}^{N} G m_i m_j \frac{(\mathbf{x}_i - \mathbf{x}_j) \cdot (\mathbf{x}_j - \mathbf{x}_i)}{|\mathbf{x}_j - \mathbf{x}_i|^3} \\ &= -\sum_{i=1}^{N} \sum_{j > i}^{N} \frac{G m_i m_j}{|\mathbf{x}_j - \mathbf{x}_i|} \\ &= \mathcal{U}, \end{aligned}$$

where $\mathcal{U}$ is the total potential energy of the system. Thus we obtain the (scalar) *Virial Theorem*:

$$2\mathcal{T} + \mathcal{U} = 0. \tag{3}$$

A stellar system for which this relation holds is said to be in *virial equilibrium.* We'll investigate later how systems get into this state. For now, we'll simply assume it to be the case.

Since the total energy is

$$E \;=\; \mathcal{T} + \mathcal{U} \quad (< 0, \;\text{note}) \, ,$$

in virial equilibrium we have

$$\mathcal{T} \;=\; -\, E, \quad \mathcal{U} \;=\; 2E.$$

2.2 Length and Time Scales

We can define some characteristic physical scales for a system in virial equilibrium (Eq. 3). For a cluster of total mass M, the *virial radius*, R_{vir}, is defined as

$$R_{vir} \equiv -\frac{GM^2}{2\mathcal{U}} = -\frac{GM^2}{4E}.$$

It represents a characteristic length scale for the cluster. It is typically comparable to the cluster half-mass radius, R_h, the radius of the sphere centered on the cluster enclosing half of the cluster's total mass. The two radii are often used interchangeably, although they are distinct physical quantities. Spitzer (1987) notes that $R_{vir} \approx 0.8R_h$ for a broad range of common cluster models.

The cluster *dynamical time scale* (or "crossing time"), t_{dyn}, is

$$\begin{aligned} t_{dyn} &\equiv \left(\frac{GM}{{R_{vir}}^3}\right)^{-1/2} = \frac{GM^{5/2}}{(-4E)^{3/2}} \\ &= 0.47 \text{ Myr} \left(\frac{M}{10^6 M_\odot}\right)^{-1/2} \left(\frac{R_{vir}}{10\text{pc}}\right)^{3/2}. \end{aligned} \tag{4}$$

The second form of this and the previous expression conveniently define R_{vir} and t_{dyn} in terms of conserved quantities.

Finally, the cluster-wide velocity dispersion $\langle v^2 \rangle$ is

$$\begin{aligned} \langle v^2 \rangle &= \frac{2\mathcal{T}}{M} = -\frac{2E}{M} \\ &= \frac{GM}{2R_{vir}} \\ &= (14.7 \text{ km/s})^2 \left(\frac{M}{10^6 M_\odot}\right) \left(\frac{R_{vir}}{10\text{pc}}\right)^{-1}. \end{aligned} \tag{5}$$

Dynamicists often write the total kinetic energy $\mathcal{T}$ in terms of the "thermodynamic" quantity kT, defined by

$$\mathcal{T} = \tfrac{3}{2} NkT$$

so

$$kT = \tfrac{1}{3}\langle m \rangle \langle v^2 \rangle = -\frac{2E}{3N} \quad \text{in virial equilibrium,} \tag{6}$$

where $\langle m \rangle = M/N$ is the mean stellar mass.

Since gravity has no preferred scale, dynamicists find it convenient to work in a system of dimensionless units such that all bulk cluster properties are of order unity. The system in widespread use, described in more detail by Heggie & Mathieu (1986), has $G = 1, M = 1$, and $E = -\frac{1}{4}$, so $R_{vir} = 1, t_{dyn} = 1$, and $\langle v^2 \rangle = \frac{1}{2}$.

3 Relaxation

The long-term evolution of a star in hydrostatic equilibrium is driven by thermal and nuclear processes that transfer energy throughout the star and generate energy in the core. In a star cluster, thermal evolution is driven by *two-body relaxation*, while energy is generated by binary interactions, as discussed later by Heggie. We now describe the process of two-body relaxation in more detail.

To a first approximation (Fig. 2), stars orbiting in a cluster move on relatively smooth orbits determined by the bulk mean-field gravitational potential of the system as a whole. However, stars occasionally experience close encounters with one another, changing their orbital parameters and transferring energy from one to the other (see Fig. 3). This is the thermalizing process that allows energy to flow around the stellar system. To estimate the time scale on which it operates, we begin by calculating the time scale for a close encounter to occur.

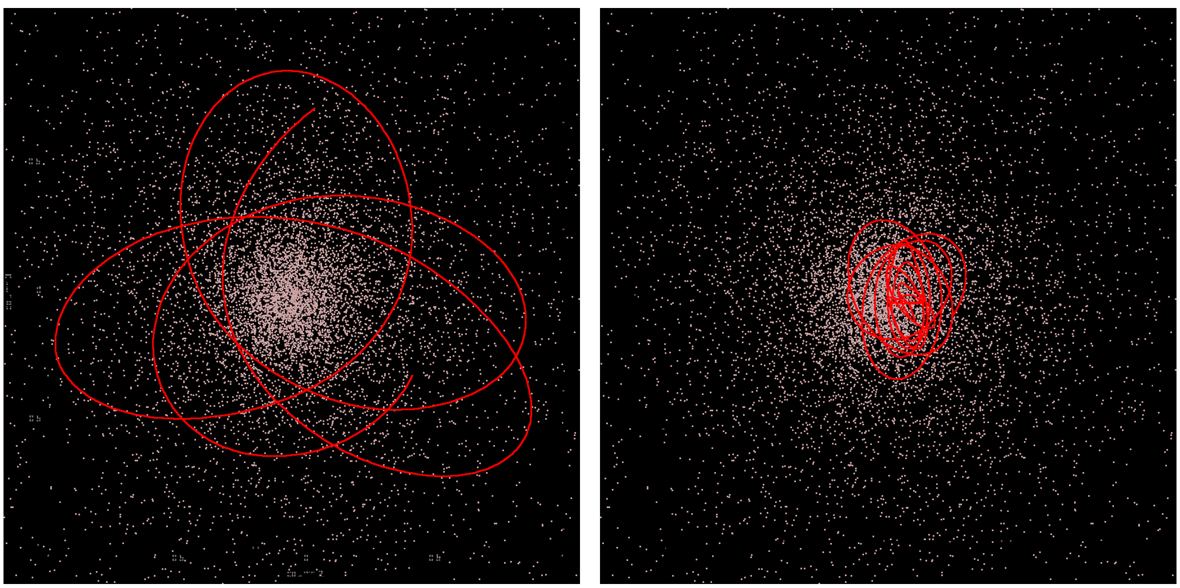

Figure 2: Two typical orbits in a 10,000-body system. Note that both orbits are smooth, although the one on the right does show one or two sharp "kinks" associated with close encounters in the dense core.

3.1 Two-body Scattering

Our basic approximation here is the assumption that encounters between stars can be treated as isolated two-body scattering events. This is permissible because, in Eq. (2), ℓ is the scale of two-body encounters, so we can talk sensibly about stellar velocities at infinity (really, at separation L) without having to worry about the large-scale motion of stars around the cluster.

The outcome of a two-body scattering encounter is well known from elementary physics. We simply quote the relevant results here. Imagine two stars of masses m_1 and m_2 ap-

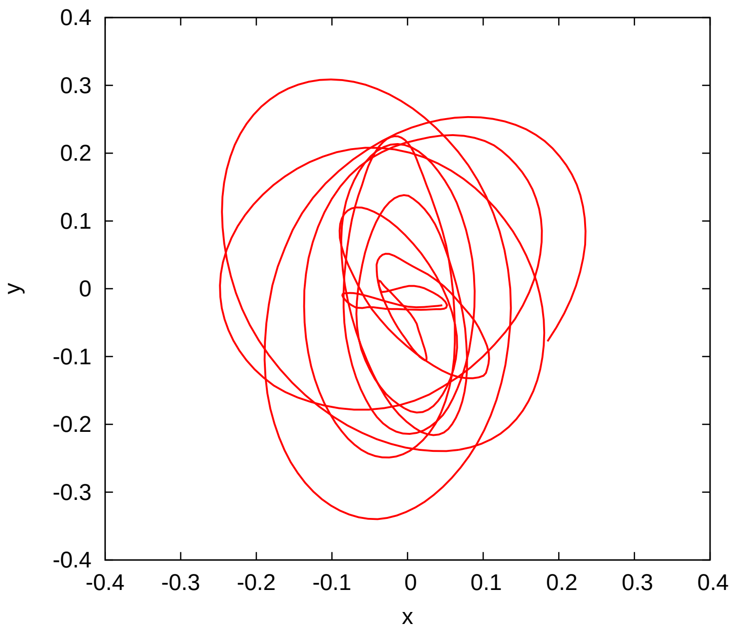

Figure 3: Enlargement of the stellar orbit in the right-hand frame of Fig. 2.

proaching one another on unbound trajectories with relative velocity at infinity v_∞ and impact parameter b (Fig. 4). The solution for the relative orbit $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ is

$$r(1 + e\cos\theta) = a(e^2 - 1),$$

where (with $m = m_1 + m_2$)

$$\begin{aligned} a &= \frac{Gm}{{v_\infty}^2} \quad \text{is the semi-major axis, and} \\ e &= \sqrt{1 + \left(\frac{bv_\infty}{Gm}\right)^2} \quad \text{is the eccentricity.} \end{aligned}$$

The deflection angle is $\psi = \pi - 2\theta_1$, where

$$\begin{aligned} \cos\theta_1 &= 1/e\,, \qquad \text{or} \\ \tan\theta_1 &= \frac{b{v_\infty}^2}{Gm}\,. \end{aligned}$$

Thus the impact parameter corresponding to a $90°$ scattering, $\psi = \pi/2$ or $\theta_1 = \pi/4$, is

$$b_{90} = \frac{Gm}{{v_\infty}^2}\,.$$

More generally, for encounters in a cluster, we have $m \sim 2\langle m\rangle$ and $\langle {v_\infty}^2\rangle \sim 2\langle v^2\rangle$, and we may write

$$b_{90} \sim \frac{G\langle m\rangle}{\langle v^2\rangle}\,. \tag{7}$$

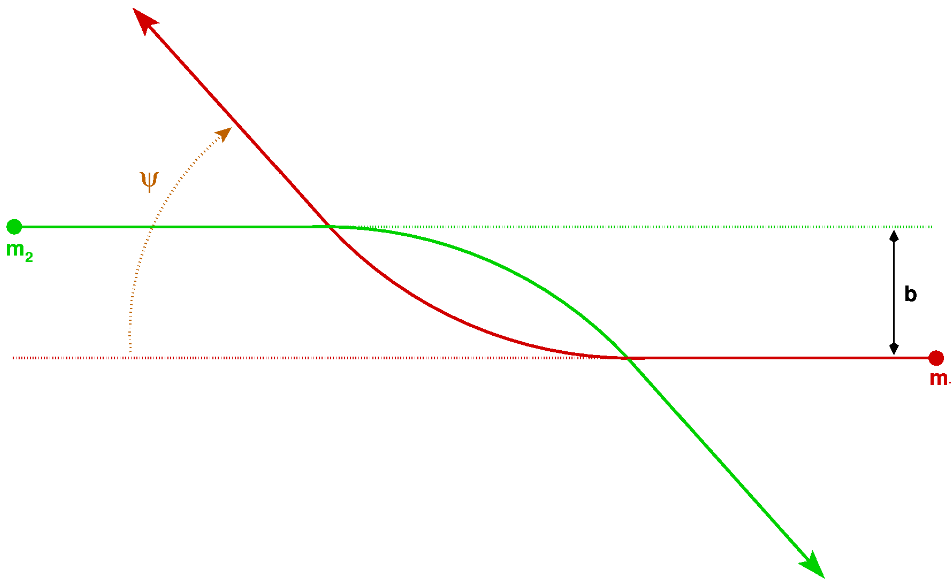

Figure 4: Two stars, of masses m_1 and m_2, approach one another with impact parameter b (and relative velocity at infinity v_∞), and are deflected by an angle ψ.

For $\langle m \rangle \sim 1 M_\odot$ and $\langle v^2 \rangle^{1/2} \sim 10$ km/s, Eq. (7) gives $b_{90} \sim 9$ AU. Combining Eqs. (5) and (7), we find $b_{90} \sim 2R_{vir}/N$.

3.2 Strong Encounters

The *strong encounter time scale*, t_s, is the time needed for a typical star to experience a 90° scattering. For a star of mass m_* moving with velocity v through a uniform field of identical stars with number density n, the cross section for a strong encounter is

$$\sigma \ = \ \pi b_{90}^2 \ = \ \frac{\pi G^2 m_*^2}{v^2} \, .$$

The time scale for a strong encounter then is

$$t_s \ = \ (n \sigma v)^{-1} \ = \ \frac{v^3}{\pi G^2 m_*^2 n} \, .$$

Replacing m_* by the mean stellar mass $\langle m \rangle$, v^2 by the mean stellar velocity dispersion $\langle v^2 \rangle$, and writing $\langle m \rangle n = \rho$, we obtain

$$t_s \ = \ \frac{\langle v^2 \rangle^{3/2}}{\pi G^2 \langle m \rangle \rho} \, . \tag{8}$$

This is the relevant time scale for discussions of interactions involving close binaries, as discussed later in Heggie's lectures.

3.3 Distant Encounters

The cross section for wide encounters, with smaller deflections $\psi \ll 1$, is much larger than that for a 90° scattering, but to estimate the cumulative effect of many small-angle deflections we must adopt a different approach. Consider again our star moving through a field of similar stars. For a single encounter with impact parameter b, the resulting velocity change transverse to the incoming velocity v is

$$\begin{aligned} \delta v_\perp &= v \sin\psi \\ &= 2v\,\frac{\tan\theta_1}{1+\tan^2\theta_1} \\ &= 2v\left(\frac{b}{b_{90}}\right)\left(1+\frac{b^2}{b_{90}^2}\right)^{-1} . \end{aligned}$$

Integrating over repeated random encounters, we expect the mean velocity change in any direction transverse to the incoming velocity to be zero, by symmetry:

$$\Delta v_\perp = \int_{-\infty}^{\infty} 2v\left(\frac{b}{b_{90}}\right)\left(1+\frac{b^2}{b_{90}^2}\right)^{-1} = 0.$$

The transverse velocity undergoes a symmetric, two-dimensional random walk, and we expect transverse velocity changes to add in quadrature, leading to a non-zero value for the mean square transverse velocity $\Delta v_\perp^2$. During a time interval δt, the number of encounters with impact parameters in the range $[b, b+db)$ is $2\pi b\, db\, nv\delta t$, so integrating over all encounters, we find

$$\begin{aligned} \Delta v_\perp^2 &= 2\pi n v \delta t \int_0^{b_{max}} b\, db\, 4v^2\left(\frac{b}{b_{90}}\right)^2\left(1+\frac{b^2}{b_{90}^2}\right)^{-2} \\ &= 8\pi n v^3 \delta t\, b_{90}^2 \int_0^{b_{max}/b_{90}} \frac{x^3\, dx}{(1+x^2)^2} \\ &\approx 8\pi\delta t\,\frac{G^2 m_*^2 n}{v}\,\ln\left(\frac{b_{max}}{b_{90}}\right) , \end{aligned}$$

where we have assumed $b_{max} \sim R_{vir} \gg b_{90}$.

We can define a *two-body relaxation time scale*, δt_r, as the time interval in the above expression corresponding to $\Delta v_\perp^2 = v^2$. Rearranging the equation and replacing all quantities by mean values, as above, we find

$$\delta t_r = \frac{\langle v^2\rangle^{3/2}}{8\pi G^2\langle m\rangle\rho\,\ln\Lambda} , \qquad (9)$$

where the "Coulomb logarithm" term (the term stemming from the almost identical development found in plasma physics) has $\Lambda = R_{vir}/b_{90} = \frac{1}{2}N$, from Eqs. (5) and (7).

There is considerable ambiguity in the above definition. For example, we could equally well have used $\Delta v_\parallel^2$ as our measure of relaxation, and our procedure neglects the distribution of relative velocities of stars in a real system. In fact, all approaches and refinements yield the same functional dependence on physical parameters as Eq. (9), but they differ in the numerical coefficient. The expression presented in Spitzer (1987), now widely adopted as a standard definition of the term, defines the relaxation time in terms of $\Delta v_{\parallel}^2$, and averages over a thermal velocity distribution—the end point (not always realized in practice) of the relaxation process. The result is

$$\begin{aligned} t_r &= \frac{0.065 \left\langle v^2 \right\rangle^{3/2}}{G^2 \langle m \rangle \rho \ln \Lambda} \qquad (10) \\ &= 3.4 \text{ Gyr} \left(\frac{\left\langle v^2 \right\rangle^{1/2}}{10 \text{km/s}} \right)^3 \left(\frac{\langle m \rangle}{M_\odot} \right)^{-1} \left(\frac{\rho}{100 M_\odot \text{pc}^{-3}} \right)^{-1} \left(\frac{\ln \Lambda}{10} \right)^{-1} . \end{aligned}$$

The precise definition of Λ is also the subject of a minor debate. Spitzer (1987) chooses $b_{max} = R_h$ and hence writes $\Lambda = 0.4N$. Giersz & Heggie (1994) calibrate the relaxation process using N-body simulations (see Lecture 3) and find $\Lambda \sim 0.1N$. For systems with a significant range of stellar masses, the effective value of Λ may be considerably smaller than even this value.

Relaxation is the process whereby a stellar system thermalizes, that is, it sets the time scale on which the velocity distribution approaches a Maxwellian. This fact is key to understanding the long-term dynamics of clusters. We note in passing that, although the analysis leading to Eq. (10) is global in nature, it is common to find this expression used as a *local* measure of the relaxation or thermalization time scale in a system.

3.4 Comparison of Time Scales

Comparing Eqs. (10) and (8), we see that

$$\frac{t_s}{t_r} \sim 5 \ln \Lambda \sim 60 \quad \text{for} \quad N \sim 10^6,$$

so distant encounters dominate over close encounters in determining the flow of energy around the system.

Spitzer (1987) defines a global relaxation time scale, often referred to as the *half-mass relaxation time*, t_{rh}, by replacing all quantities in Eq. (10) with their system-wide averages,

$$\begin{aligned} \langle v^2 \rangle &\rightarrow \frac{GM}{2R_{vir}} \\ \rho &\rightarrow \frac{3M}{8\pi {R_h}^3} \\ \langle m \rangle &\rightarrow \frac{M}{N}, \end{aligned}$$

obtaining

$$\begin{aligned} t_{rh} &= \frac{0.138 \, N {R_h}^{3/2}}{G^{1/2} M^{1/2} \ln \Lambda} \qquad (11) \\ &= 6.5 \text{ Gyr} \left(\frac{N}{10^6} \right) \left(\frac{M}{10^6 M_\odot} \right)^{-1/2} \left(\frac{R_h}{10 \text{ pc}^{-3}} \right)^{3/2} \left(\frac{\ln \Lambda}{10} \right)^{-1} . \end{aligned}$$

Hence, from Eqs. (4) and (11), we have

$$\frac{t_{rh}}{t_{dyn}} \sim \frac{N}{5 \ln \Lambda},$$

and we see that relaxation is a slow process relative to the dynamical time for all but the smallest systems.

Relaxation is driven by the granularity in a stellar system—that is, the statistical departure of the system from its mean-field average. As the number of stars in the system increases, the granularity decreases and relaxation slows relative to the crossing time. Systems in which the relaxation time is shorter than the time scale of interest—for example, the Hubble time—and hence for which relaxation is an important evolutionary process, are often referred to as *collisional* systems. (Note that this dynamical terminology does not necessarily have anything to do with the possibility of actual stellar collisions, such as those discussed later by Davies.) Systems in which the relaxation time is longer than any other time scale of interest are called *collisionless*. Open and globular clusters, as well as some galactic nuclei (including the nucleus of the Milky Way galaxy), and some dwarf galaxies, are collisional. The disks and halos of large spiral and elliptical galaxies are collisionless.

4 Particle and Continuum Models

Modeling methods for star clusters fall fairly naturally into two categories. The first simply treats the system as a discrete collection of N interacting particles, as the above presentation has done, and focuses on following the motion of each particle under the combined gravitational influence of all others. We will present and discuss some details of this approach in Lecture 3.

The second category starts from an idealized continuum description of the stellar system—comparable to the fluid description of stellar structure—and includes relaxation and other effects as needed. Such methods, although approximate, are much faster than the particle approach, and continue to contribute greatly to our understanding of cluster evolution. They include gas-sphere, Fokker–Planck, and Monte Carlo descriptions of cluster dynamics.

Continuum descriptions of cluster dynamics generally begin with the collisionless Boltzmann equation, which describes the flow of a collisionless fluid in a 6-dimensional phase space. If $f(\mathbf{x}, \mathbf{v}, t)$ is the distribution function of the fluid—that is, the number of particles in 6-d volume $d^3x\, d^3v$ is $f(\mathbf{x}, \mathbf{v}, t)\, d^3x\, d^3v$—the Boltzmann equation is a continuity equation in phase space:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{12}$$

The gravitational nature of the system is included via $\mathbf{a} = -\nabla\phi$. The "0" on the right-hand side of Eq. (12) means that particles interact only via their bulk gravity—relaxation effects are explicitly excluded. Incorporation of relaxation into the equation in essence entails adding a collisional term

$$\Gamma[f] \equiv \left.\frac{df}{dt}\right|_c$$

on the right.

Two continuum approaches deserve mention here. Both simplify the geometry of the system to spherical symmetry. *Gas-sphere methods* integrate the Boltzmann equation locally over velocity, introducing successive moments of f, $\rho(\mathbf{x}) = \int f\,d^3v$, $\langle v^2\rangle(\mathbf{x}) = \int f v^2\,d^3v$, etc., and recasting the equation as a set of coupled spatial differential equations closely resembling the equations of stellar structure and evolution, with relaxation playing the role of conductivity in transporting energy from place to place. Specifically, as described by Lynden-Bell & Eggleton (1980), the energy transport equation

$$\frac{\partial T}{\partial r} \;=\; -\,\frac{L}{4\pi\kappa r^2}\,,$$

where $T(r)$ is temperature, $L(r)$ is luminosity, and κ is the thermal conductivity, becomes, in the stellar dynamical case

$$\frac{\partial}{\partial r}\left(\tfrac{1}{2}\langle v^2\rangle\right) \;=\; -\,\frac{L}{4\pi\kappa r^2}\,.$$

In the latter form, L is the conductive luminosity, $\langle v^2\rangle$ is related to T by Eq. (6), and the "conductivity" is

$$\kappa \;=\; \frac{C\rho\lambda^2}{t_r}\,,$$

where $C \sim 1$ and λ is the local Jeans length, defined by $\lambda^2 = \pi\langle v^2\rangle/3G\rho$—a local analog to the global virial radius (see Eq. 5).

Alternatively, we can add to the right-hand side of the Boltzmann Equation (12) a term Γ describing the effect of two-body scattering in redistributing stars from one part of phase space to another. Following Binney & Tremaine (2008) and writing for brevity $\mathbf{w} = (\mathbf{x}, \mathbf{v})$, we have

$$\Gamma[f] \;=\; \int d^6\Delta w\,\left[\Psi(\mathbf{w}-\boldsymbol{\Delta}\mathbf{w},\boldsymbol{\Delta}\mathbf{w})f(\mathbf{w}-\boldsymbol{\Delta}\mathbf{w}) - \Psi(\mathbf{w},\boldsymbol{\Delta}\mathbf{w})f(\mathbf{w})\right], \qquad (13)$$

where $\Psi(\mathbf{w},\boldsymbol{\Delta}\mathbf{w})d^6\Delta w\Delta t$ is the probability that a star with phase space coordinates $\mathbf{w}$ is scattered into the volume $d^6\Delta w$ around $\mathbf{w}+\boldsymbol{\Delta}\mathbf{w}$ in time Δt. In principle, Ψ can be determined directly from the considerations presented in §3.1. If, as we have argued, relaxation is driven predominantly by weak encounters, then $|\boldsymbol{\Delta}\mathbf{w}|$ is small and we can approximate the integrand in Eq. (13) by the first two terms of a Taylor series:

$$\begin{aligned}\Psi(\mathbf{w}-\boldsymbol{\Delta}\mathbf{w},\boldsymbol{\Delta}\mathbf{w})&f(\mathbf{w}-\boldsymbol{\Delta}\mathbf{w}) - \Psi(\mathbf{w},\boldsymbol{\Delta}\mathbf{w})f(\mathbf{w})\\ &\approx\; -\sum_{i=1}^{6}\Delta w_i\frac{\partial}{\partial w_i}\left[\Psi(\mathbf{w},\boldsymbol{\Delta}\mathbf{w})f(\mathbf{w})\right]\\ &\quad+\tfrac{1}{2}\sum_{i,j=1}^{6}\Delta w_i\Delta w_j\frac{\partial^2}{\partial w_i\partial w_j}\left[\Psi(\mathbf{w},\boldsymbol{\Delta}\mathbf{w})f(\mathbf{w})\right].\end{aligned} \qquad (14)$$

This is the *Fokker-Planck approximation.* Carrying out the integrals over $d^6\Delta w$ in Eq. (14) results in the *Fokker-Planck equation*

$$\begin{aligned}\frac{df}{dt} \;&=\; -\sum_{i=1}^{6}\frac{\partial}{\partial w_i}\left[f(\mathbf{w})\langle w_i\rangle\right]\\ &\quad+\tfrac{1}{2}\sum_{i,j=1}^{6}\frac{\partial^2}{\partial w_i\partial w_j}\left[f(\mathbf{w})\langle w_iw_j\rangle)\right],\end{aligned} \qquad (15)$$

where

$$
\begin{aligned}
\langle w_i \rangle &= \int d^6\Delta w \, \Delta w_i \Psi(\mathbf{w}, \mathbf{\Delta w}) \, , \\
\langle w_i w_j \rangle &= \int d^6\Delta w \, \Delta w_i \Delta w_j \Psi(\mathbf{w}, \mathbf{\Delta w}) \, .
\end{aligned}
$$

The quantities $\langle w_i \rangle$ and $\langle w_i w_j \rangle$ are diffusion coefficients that depend only on local phase-space coordinates. They are significantly simplified when interactions can be viewed as local, so that $\mathbf{\Delta x} = 0$ and Eq. (15) becomes

$$
\begin{aligned}
\frac{df}{dt} &= -\sum_{i=1}^{3} \frac{\partial}{\partial v_i} \left[f(\mathbf{w}) \langle v_i \rangle \right] \\
&\quad + \tfrac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2}{\partial v_i \partial v_j} \left[f(\mathbf{w}) \langle v_i v_j \rangle) \right] .
\end{aligned}
$$

The velocity-space Fokker-Planck equation has proved a very useful tool for numerical studies of cluster dynamics. A Monte Carlo realization of this approach developed by Spitzer and coworkers (see Spitzer 1987 for references) has been instrumental in furthering our understanding of cluster evolution.

It can be shown that the distribution function f of a collisionless, stationary system is a function of conserved quantities only. For a spherically symmetric system, those quantities are the energy E and the angular momentum J. In that case, the Fokker-Planck equation can be simplified by orbit averaging all components:

$$
\overline{Q} = \frac{1}{P} \oint Q(r; E, J) \, \frac{dr}{v_r} \, ,
$$

where v_r is the radial velocity and $P(E, J)$ is the orbital period. The result, with $N(E, J, t)$ the density of stars in $E - J$ space, is the *orbit averaged Fokker-Planck Equation*:

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= -\frac{\partial}{\partial E} \left[N \overline{\langle \Delta E \rangle} \right] - \frac{\partial}{\partial J} \left[N \overline{\langle \Delta J \rangle} \right] + \tfrac{1}{2} \frac{\partial^2}{\partial E^2} \left[N \overline{\langle (\Delta E)^2 \rangle} \right] \\
&\quad + \frac{\partial^2}{\partial E \partial J} \left[N \overline{\langle \Delta E \Delta J^2 \rangle} \right] + \tfrac{1}{2} \frac{\partial^2}{\partial J^2} \left[N \overline{\langle (\Delta J)^2 \rangle} \right] \, .
\end{aligned}
$$

It has formed the basis for many landmark studies of globular cluster dynamics.