A short introduction to cosmological formulae

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This short introduction is extracted from a paper on cosmological models with a dark energy component (Cappi 2001). For an on–line implementation of cosmological formulae see www.bo.astro.it/∼cappi/cosmotools.

The basic equations for calculating distances in relativistic cosmology are briefly resumed (e.g. Peebles 1993; Coles & Lucchin 1996, Peacock 1999; see also Hogg 1999). Assuming that the universe is homogeneous and isotropic, we have the Robertson–Walker metric:

$$ds^2 = c^2 dt^2 - R^2(t) \left[dr^2 + S_k^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{1}$$

where $S_k(r) = r$ for $k = 0$, $S_k(r) = \sin(r)$ for $k = 1$ and $S_k(r) = \sinh(r)$ for $k = -1$, and from Einstein's field equations the Friedmann equations are obtained:

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{kc^2}{R^2} \tag{2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \sum_i \rho_i (1 + 3w_i) \tag{3}$$

where ρ_i is the energy density of component i and $w_i = P_i/\rho_i c^2$ is the corresponding equation of state.

The equation of state of matter is $w = 0$ while for radiation (and relativistic matter) $w = 1/3$. For a dark energy component $w < 0$: $w = -1$ for the cosmological constant, $-1 < w < 0$ for "quintessence" and $w < -1$ for "phantom" energy.

Defining the ratios $\Omega_i \equiv \rho_i/\rho_c$, where $\rho_c \equiv 3H^2/8\pi G$ and $\Omega_k \equiv -kc^2/(HR)^2$, we can rewrite equation (2) as: $\Omega_{tot} = \sum_i \Omega_i = 1 - \Omega_k$.

Moreover, dividing equation (3) by equation (2) we obtain the general expression for Sandage's deceleration parameter $q = -\ddot{R}R/\dot{R}^2 = \frac{1}{2}(1-\Omega_k) + \frac{3}{2}\sum_i \Omega_i w_i$.

The comoving coordinate r can be written as a function of Ω_i:

$$r = \sqrt{|\Omega_k|} \int_0^z \frac{dz'}{E(z')} \tag{4}$$

where:

$$E(z) = \sqrt{\Omega_k(1+z)^2 + \sum_i \Omega_i (1+z)^{3(1+w_i)}} \tag{5}$$

The comoving distance[1] is therefore given by:

$$d_c(z) = R_0 S_k(r) = \frac{c}{H_0} \frac{1}{|\Omega_k|^{1/2}} S_k(r) \tag{6}$$

where, as usual, the subscript 0 indicates the value at the present epoch. The luminosity distance d_L is simply given by $d_L = d_c(1+z)$, while the angular distance is $d_A(z) = d_c/(1+z)$.

The lookback time is also given by an analogous integral:

$$t(z) = \frac{1}{H_0} \int_0^z \frac{1}{(1+z')E(z')} \tag{7}$$

[1] $d_c(z)$ corresponds to the transverse comoving distance D_M as defined by Hogg (1999).

References

[1] Cappi A., 2001, ApL&C 40, 161 (astro-ph/0105382)

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