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Analytical Properties of the $R^{1/m}$ Law

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Abstract. In this paper we describe some analytical properties of the $R^{1/m}$ law proposed by Sersic (1968) to categorize the photometric profiles of elliptical galaxies. In particular, we present the full asymptotic expansion for the dimensionless scale factor $b(m)$ that is introduced when referring the profile to the standard effective radius. Surprisingly, our asymptotic analysis turns out to be useful even for values of m as low as unity, thus providing a unified analytical tool for observational and theoretical investigations based on the $R^{1/m}$ law for the entire range of interesting photometric profiles, from spiral to elliptical galaxies. The systematic asymptotic analysis provided here also allows us to clarify the value and the limitations of the power law R^{-2} often used in the past as a natural representation of the surface brightness profiles of elliptical galaxies.

1. Introduction

After its introduction as a generalization of the $R^{1/4}$ law (de Vaucouleurs 1948), the $R^{1/m}$ law (Sersic 1968) has been widely used in observational and theoretical investigations (see, e.g., Ciotti & Bertin 1999, and references therein, hereafter CB). According to such law, the surface brightness profile is given by

$$I(R) = I_0 e^{-b\eta^{1/m}}, \quad m > 0, \tag{1}$$

where $\eta = R/R_e$, and b is a dimensionless constant such that R_e is the effective (half–luminosity) radius. The projected luminosity inside R is given by

$$L(R) = 2\pi \int_0^R I(R')R'dR' = I_0 R_e^2 \frac{2\pi m}{b^{2m}} \gamma(2m, b\eta^{1/m}), \tag{2}$$

where $\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt$ is the (left) incomplete gamma function. The total luminosity is then given by

$$L = I_0 R_e^2 \frac{2\pi m}{b^{2m}} \Gamma(2m), \tag{3}$$

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where $\Gamma(\alpha) = \gamma(\alpha, \infty)$ is the complete gamma function. It follows that $b(m)$ is the solution of the following equation:

$$\gamma(2m, b) = \frac{\Gamma(2m)}{2}. \tag{4}$$

2. Asymptotic expansion

Unfortunately, eq. (4) cannot be solved in explicit, closed form, and so it is usually solved numerically. This is inconvenient for a number of observational and theoretical applications. At least three interpolation formulae for $b(m)$ have been given in the literature, namely $b \simeq 1.9992m - 0.3271$ by Capaccioli (1989), $b \simeq 2m - 0.324$ by Ciotti (1991), and $b \simeq 2m - 1/3 + 0.009876/m$ by Prugniel & Simien (1997). These expressions provide an accurate fit in the range $0.5 \leq m \leq 10$; curiously, their leading term is *linear* in m, with a slope very close to 2. In CB it is proved that this behavior results from a general property of the gamma function. In particular, it turns out that the first terms of the asymptotic expansion of $b(m)$ as implicitly given by eq. (4) are

$$b(m) \sim 2m - \frac{1}{3} + \frac{4}{405m} + \frac{46}{25515m^2} + \mathrm{O}(m^{-3}). \tag{5}$$

Equation (5) now clearly explains the value of the interpolation formulae found earlier. Note that $4/405 = 0.009876\ldots$. Of course, the asymptotic analysis provided in CB would allow us to give explicitly any higher order term if so desired. In CB it is shown that this expansion, even when truncated to the first four terms as in eq. (5), performs much better than the interpolation formulae cited above, even for m values as low as unity, with relative errors smaller than 10^{-6}.

The use of this simple formula is thus recommended both in theoretical and observational investigations based on the Sersic law. As a simple application, in CB the leading term of the asymptotic expansion of the total luminosity, of the central potential, and of the surface brightness profile associated with the $\mathrm{R}^{1/m}$ law are obtained in explicit, analytical form. In particular, the connection of eq. (1) with the simple power law R^{-2}, often used in the past to fit the photometric profiles of elliptical galaxies, is brought out explicitly.

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