Astron. Astrophys. 249, 99–106 (1991)

ASTRONOMY AND ASTROPHYSICS

Stellar systems following the $R^{1/m}$ luminosity law

L. Ciotti

Dipartimento di Astronomia, Università di Bologna, Via Zamboni 33, I-40126 Bologna, Italy

Received February 11, accepted March 26, 1991

Abstract. In this paper I study the structural and dynamical properties of isotropic spherical galaxies with surface luminosity profile following the $R^{1/m}$-law. For these models the luminosity density, the potential, the velocity dispersion and other related deprojected properties are presented in detail, together with analytical expressions for their asymptotic behavior at small and large radii. The parameter m for the computed models covers the range [2, 10]. Moreover, I construct and study the *dimensionless* self-consistently generated distribution function $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$, and the normalized differential energy distribution $N_m(\widetilde{\mathscr{E}}_m)$, where $\widetilde{\mathscr{E}}_m = \mathscr{E}/\psi_m(0)$ is the *relative binding energy* $\mathscr{E}$, normalized to the central value of the potential of the model, $\psi_m(0)$. The two functions $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ and $N_m(\widetilde{\mathscr{E}}_m)$ are calculated under the assumptions that the velocity dispersion tensor is isotropic and that the mass-to-light ratio (M/L) is constant. The main results are the following:
1) For any m, both $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ and its derivative are positive for bound stars, i.e. for positive values of $\widetilde{\mathscr{E}}_m$. This means that the investigated models are physically admissible and stable.
2) For any m, a large interval of width $\Delta\widetilde{\mathscr{E}}_m$ exists over which $N_m(\widetilde{\mathscr{E}}_m)$ is very well described by the Boltzmann formula $N_B(\mathscr{E}) = N_o exp(-\beta\widetilde{\mathscr{E}}_m)$, with $\beta > 0$. Moreover, $\Delta\widetilde{\mathscr{E}}_m$ is nearly independent of m, being $\Delta\widetilde{\mathscr{E}}_m \simeq 0.5$.
3) Finally, for any m, I compute $\Delta M_m/M_m$, i.e. the fraction of stellar mass within the Boltzmann range $\Delta\widetilde{\mathscr{E}}_m$, normalized to the total mass of the model, M_m. I find that $\Delta M_m/M_m$ depends in a very sensitive way on the parameter m, having a maximum for $m = 4$, i.e. for the de Vaucouleurs law.

Key words: galaxies: elliptical – galaxies: kinematics and dynamics – galaxies: structure of

1. Introduction

The $R^{1/4}$-law was introduced by de Vaucouleurs (1948) to describe the projected luminosity density (or surface brightness) $I(R)$ of elliptical galaxies (see Eq. (1) below, with $m = 4$). It has no *free parameters* and depends on two well defined *physical scales*: a characteristic linear scale, R_e, and a luminosity density factor, I_o. The surface brightness of elliptical galaxies is well fitted by the $R^{1/4}$-law: Kormendy (1977) has shown that this law, when compared to other empirical laws often used, provides the best fit for dynamically unperturbed elliptical galaxies. De Vaucouleurs and Capaccioli (1979) have shown that the surface brightness of NGC 3379 (an E1 galaxy) is fitted (with residuals smaller than 0.08 magnitudes) over 10 magnitudes and more than two decades in radius by such a law, and new fits made by Capaccioli *et al.* (1990) based on fresh data are in excellent agreement with the previously cited results. However, Makino *et al.* (1990) stated that in the range of radii usually investigated in the observations, $R^{1/m}$-law models are practically indistinguishable from the de Vaucouleurs law, for $3 \leq m \leq 10$, and then that the universality of the $R^{1/4}$-law is only apparent. In any case, given the empirical success of the $R^{1/4}$-law, much theoretical work has been devoted to study the dynamical properties of stellar dynamical models characterized by such a law (for constant M/L ratio). Useful numerical tables of the spatial values for density and for other related deprojected properties were calculated by Poveda *et al.* (1960) and Young (1976). Mellier and Mathez (1987) presented an analytical fit to the numerical values of the luminosity density, whose projection gives the $R^{1/4}$-law: in the range $0.1 \leq R/R_e \leq 100$ the logarithmic error is less than 0.01.

On the physical side, by means of N-body simulations, dissipationless collapses have been shown to produce systems following the de Vaucouleurs law very closely (see, e.g. van Albada 1982, Villumsen 1984, Aguilar and Merritt 1990, Londrillo *et al.* 1991). In these simulations the *differential energy distribution* $dM/d\mathscr{E}$, i.e. the star mass fraction for unit energy, is usually plotted as indicator of the relaxation of the system (May and van Albada 1984). Bertin and Stiavelli, using a *selection criterion* derived mostly from the results of N-body simulations of collisionless collapse (Bertin and Stiavelli 1984, 1987; Stiavelli and Bertin 1985; Bertin, Saglia and Stiavelli 1988), were indeed able to construct a distribution function which, for a sufficiently high central concentration of the system, recovers naturally the de Vaucouleurs law. Binney (1982, hereafter B82), assuming a constant M/L ratio, spherical symmetry, and isotropy of the velocity dispersion tensor, showed that the distribution function of the $R^{1/4}$-law is positive for any admissible value of $\mathscr{E}$, and that its differential energy distribution is well approximated by the Boltzmann formula with positive "temperature" over the radial range where the de Vaucouleurs law has a secure observational support. This exponential behavior is substantially confirmed by N-body simulations. In contrast, Bertin and Stiavelli (1989) argued that "*the properties of the energy distribution are more a measure of the core concentration of the system than a consequence of the projected luminosity being in agreement with the de Vaucouleurs law*".

This discussion shows that a number of empirical and theoretical studies point to the importance of the $R^{1/4}$-law and that at the same time it is not clear whether other laws, which for simplicity we parametrize as $R^{1/m}$-laws, could equally fit the ob-

servational data and the theoretical framework that has been recently developed to explain the photometric profiles of elliptical galaxies. Therefore, in this paper I investigate the dynamical properties of $R^{1/m}$ models focusing on the following questions:
1) Is the property of an exponential differential energy distribution a characteristic of stellar systems following the de Vaucouleurs law or is it a more common feature of sufficiently concentrated stellar systems, as $R^{1/m}$-law models with $m > 1$?
2) If this is the case, is there a significant physical property that characterizes the de Vaucouleurs law among all the $R^{1/m}$-law models?

In order to answer these questions, I carry out an extensive investigation of the main properties characterizing stellar systems following the $R^{1/m}$ luminosity law. In Sect. 2 I present the basic, photometric, spatial and dynamical properties of the $R^{1/m}$-law models derived under the assumption of spherical symmetry, constant mass–to–light ratio M/L, and isotropy of the velocity dispersion tensor; thus I compute some properties of the assumed surface brightness profile, the spatial luminosity and density, the potential, the spatial and line–of–sight velocity dispersion for $2 \leq m \leq 10$. In Sect. 3 I present the dimensionless distribution function $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$, and the normalized differential energy distribution $N_m(\widetilde{\mathscr{E}}_m)$, associated with these models. A characteristic property of the distribution N_4 is identified. Finally, in Sect. 4 I summarize the results. In Appendix A the asymptotic formulae for the luminosity density and other properties are given, and in Appendix B the central behavior of the distributions $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ and $N_m(\widetilde{\mathscr{E}}_m)$ is analytically deduced for models with $m > 1$.

2. The $R^{1/m}$-law models

2.1. *The photometric (projected) properties*

The $R^{1/m}$-law models can be defined as a one-parameter family of stationary, spherical stellar systems, with surface brightness profile generalizing the de Vaucouleurs law in the following way:

$$\ln I(R) = \ln I_o - b\eta^{1/m}. \tag{1}$$

I_o is the central surface brightness and $\eta = R/R_e$, where R_e is the *effective radius* (i.e. the projected radius inside which the projected luminosity, given by Eq. (2), equals half of the total luminosity). The defining parameter is m, a positive real number, while b is a dimensionless parameter whose value is determined by the definition of R_e, as shown in Eq. (5); thus b can be seen as a function of m: for $m = 4$, $b(4) = 7.66924944$. In Fig. 1 (left) the family of curves $\Delta\mu(\eta) = \mu(0) - \mu(\eta)$, where $\mu(\eta) = -2.5 \log_{10} I(R)$, is shown for $2 \leq m \leq 10$.

The projected luminosity, i.e. the integrated luminosity inside the cylinder of radius R, is given by

$$LP(R) = 2\pi \int_0^R I(R')R'dR'. \tag{2}$$

We can write $LP_m(R) = I_o R_e^2 \times \widetilde{LP}_m(\eta)$, where the dimensionless function $\widetilde{LP}_m(\eta)$ is given by:

$$\widetilde{LP}_m(\eta) = \frac{2\pi m}{b^{2m}} \gamma(b\eta^{1/m}, 2m). \tag{3}$$

Here γ is the incomplete gamma function (see, e.g., Erdély, Magnus, Oberhettinger, Tricomi 1953, hereafter EMOT53, vol. II, p. 153). The expression for the total luminosity L_m is obtained by taking the limit of Eq. (3) for $\eta \to \infty$, and the total mass of the model is then immediately given by $M_m = (M/L) \times L_m$. We have $L_m = I_o R_e^2 \times \widetilde{L}_m$, where

$$\widetilde{L}_m = \frac{2\pi m}{b^{2m}} \Gamma(2m), \tag{4}$$

and Γ is the complete gamma function (EMOT53, vol. I, p. 1). The values of $\log_{10} \widetilde{L}_m$ are listed in Tab.1, and the family of the normalized projected luminosities, defined as $\widetilde{LP}_m(\eta)/\widetilde{L}_m$, is shown in Fig. 1 (right).

From the definition of R_e and Eqs. [(3),(4)], $b(m)$ can be obtained as the solution of the following equation:

$$\gamma(b, 2m) = \frac{\Gamma(2m)}{2}. \tag{5}$$

The function $b = b(m)$ is very well fitted by the linear interpolation $b(m) = 2m - 0.324$, for $0.5 \leq m \leq 10$, with relative errors smaller than 0.001. The exact values of $b(m)$ are recorded in Tab.1 for $2 \leq m \leq 10$.

Table 1. The dimensionless parameters $b(m)$, $\log_{10} \widetilde{L}_m$, and $\log_{10} \widetilde{\psi}_m(0)$, for $2 \leq m \leq 10$

m	$b(m)$	$\log_{10} \widetilde{L}_m$	$\log_{10} \widetilde{\psi}_m(0)$
2	3.67206075	-0.38227831	-0.22672973
3	5.67016119	-1.16709006	-0.88057497
4	7.66924944	-1.97535251	-1.5567402
5	9.66871461	-2.79677451	-2.2456025
6	11.6683632	-3.62663241	-2.9426671
7	13.6681146	-4.46236234	-3.6454698
8	15.6679295	-5.30241629	-4.3525124
9	17.6677864	-6.14578764	-5.0628162
10	19.6676724	-6.99178484	-5.7757066

2.2. *The spatial (deprojected) properties*

The most important deprojected quantity is the *luminosity density* ν, which is related to the mass density via $\rho(r) = (M/L)\nu(r)$. As usual, r is the spatial (or deprojected) radius. I assume a constant M/L ratio, so that the fundamental quantities (mass inside r, potential, velocity dispersion, etc.) depend only on the luminosity density $\nu(r)$, which is related to the surface brightness profile by an Abel integral equation (see, e.g., Binney and Tremaine 1987, hereafter BT87):

$$\nu(r) = -\frac{1}{\pi} \left[\int_r^\infty \frac{dI}{dR} \frac{dR}{\sqrt{R^2 - r^2}} - \lim_{R \to \infty} \frac{I(R)}{\sqrt{R^2 - r^2}} \right]. \tag{6}$$

The second term in the r.h.s. of Eq. (6) is zero for any positive value of m. The luminosity density ν_m can be expressed as $\nu_m(r) = (I_o/R_e) \times \widetilde{\nu}_m(s)$, with

$$\widetilde{\nu}_m(s) = \frac{b^m \alpha^{1-m}}{\pi} \mathscr{C}_m^0(\alpha), \tag{7}$$

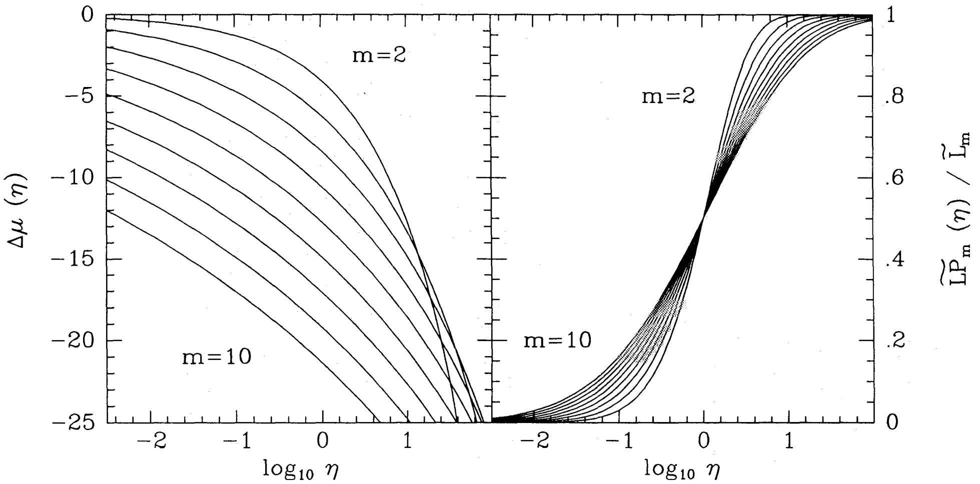

Fig. 1. Left panel: the family of curves $\Delta\mu(\eta) = \mu(0) - \mu(\eta)$ for the computed models. The upper curve is relative to $m = 2$ and the lower to $m = 10$. For any two values of the parameter m, the η of the intersection of the corresponding $\Delta\mu$ is given by $[b(m)/b(m')]^{mm'/(m-m')}$. Right panel: the normalized projected luminosities, $\widetilde{LP}_m(\eta)/\widetilde{L}_m$, for $2 \le m \le 10$. In the abscissae scale, $\eta = R/R_e$

where the function $\mathscr{C}_m^\beta(\alpha)$ is defined in Appendix A. Here and in the following $\alpha = bs^{1/m}$ is the *reduced radial coordinate*, and $s = r/R_e$. The asymptotic behavior of $\widetilde{\nu}_m$ for $r \to \infty$ is given by:

$$\widetilde{\nu}_m(s) \sim \sqrt{\frac{b}{2\pi m}} \exp(-bs^{1/m}) s^{(1-2m)/2m}, \tag{8}$$

and, for $r \to 0$ and $m > 1$, by:

$$\widetilde{\nu}_m(s) \sim \frac{B[1/2, (m-1)/2m]}{2mb^{m-1}} \exp(-bs^{1/m}) s^{(1-m)/m}, \tag{9}$$

where $B(x, y)$ is the complete beta function (EMOT53, vol. I, p. 9). The derivation of Eqs. [(7)–(9)] is given in Appendix A. For the particular case of $m = 4$, the relevant exponent at the origin is $-3/4$, as reported by Mellier and Mathez (1987), and the numerical values of $\widetilde{\nu}_4$ obtained are in full agreement with the tables of Young (1976). For $m \neq 4$, the accuracy of the computation is tested by comparing the values given by Eq. (1) with those obtained by projecting the luminosity density calculated numerically. In Fig. 2 (left) the family of $\log_{10} \widetilde{\nu}_m(s)$ for $2 \le m \le 10$ is shown. It should be noted that for $m > 1$ the density diverges at the origin as $r^{(1-m)/m}$, and therefore the divergence of ρ is worse for higher-m models.

From the luminosity density we can calculate the integrated spatial luminosity, $L(r) = 4\pi \int_0^r \nu(r') r'^2 dr'$. For the $R^{1/m}$-law models we have $L_m(r) = I_o R_e^2 \times \widetilde{L}_m(s)$. In Fig. 2 (right) the family of the normalized spatial luminosity $\widetilde{L}_m(s)/\widetilde{L}_m$ is shown for $2 \le m \le 10$. From Fig. 2 it is evident that, if one defines s_h as the spatial half–light radius of a model normalized to its effective radius R_e, this number is practically independent of m. Exact values of s_h are reported in Tab. 2.

2.3. The dynamical properties

The fundamental property which defines the dynamical behavior of the $R^{1/m}$-law models is their associated potential. I will refer to the positive function $\psi_m(r) = G(M/L) I_o R_e \times \widetilde{\psi}_m(s)$, where G is the gravitational constant and, from the Gauss theorem,

$$\widetilde{\psi}_m(s) = \int_s^\infty \frac{\widetilde{L}_m(x)}{x^2} dx. \tag{10}$$

The integral in Eq. (10) cannot be solved in terms of elementary functions, but for $r = 0$ one can use the following general relation for spherical systems:

$$\psi(0) = -4G(M/L) \int_0^\infty \frac{dI}{dR} R dR, \tag{11}$$

from which the value of the central potential is found to be:

$$\widetilde{\psi}_m(0) = \frac{4\Gamma(m+1)}{b^m}. \tag{12}$$

In Tab. 1 I list the values of the quantity $\log_{10} \widetilde{\psi}_m(0)$, and in Fig. 3 (left) I show the family of the normalized potentials $\widetilde{\psi}_m(s)/\widetilde{\psi}_m(0)$. It is easy to note that models with low m have a potential well much flatter than models with high m, in agreement with the central behavior of the density, as reported in Eq. (B1) for $x = 0$.

Table 2. The dimensionless parameters s_h, s_h/s_g, $\beta(m)$ and $\Delta M_m/M_m$, for $2 \le m \le 10$

m	s_h	s_h/s_G	$\beta(m)$	$\Delta M_m/M_m$
2	1.3386	0.4137	1.52	0.50
3	1.3428	0.4287	1.88	0.54
4	1.3490	0.4536	2.26	0.56
5	1.3490	0.4854	2.51	0.27
6	1.3490	0.5259	2.78	0.19
7	1.3552	0.5726	3.00	0.12
8	1.3552	0.6268	3.19	0.09
9	1.3552	0.6908	3.39	0.08
10	1.3552	0.7624	3.57	0.05

As far as the velocity dispersion is concerned, under the assumption of isotropic pressure (see Sect. 3), and of spherical symmetry, the Jeans equation becomes $d(\nu\sigma^2)/dr = \nu d\psi/dr$, with

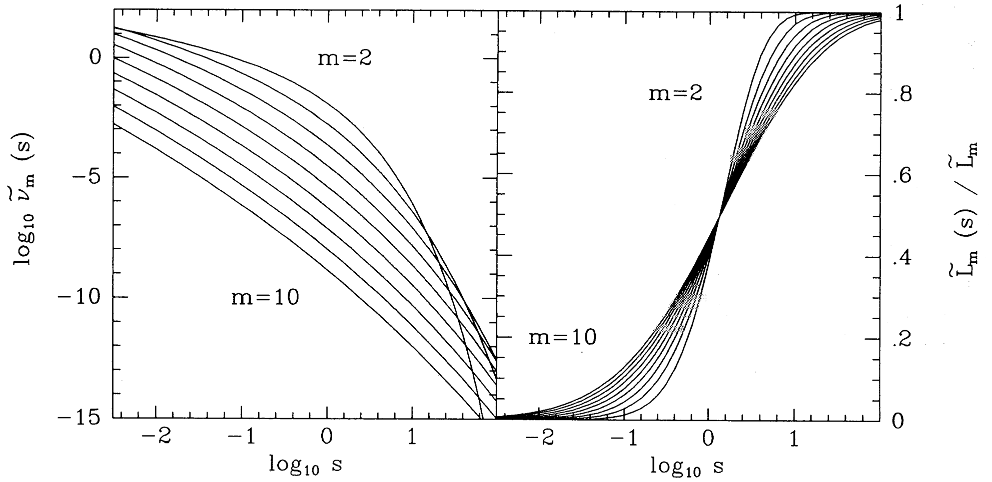

Fig. 2. Left panel: the luminosity densities, $\log_{10} \widetilde{\nu}_m(s)$, for $2 \le m \le 10$. Right panel: the normalized spatial luminosities, $\widetilde{L}_m(s)/\widetilde{L}_m$, for the same range of m. In the abscissae scale, $s = r/R_e$

the usual boundary condition given by $\nu\sigma^2 \to 0$ for $r \to \infty$. This yields:

$$\widetilde{\sigma}_m^2(s) = \frac{1}{\widetilde{\nu}_m(s)} \int_s^\infty \frac{\widetilde{L}_m(x)}{x^2} \widetilde{\nu}_m(x) dx, \tag{13}$$

where $\sigma_m^2(r) = G(M/L)I_o R_e \times \widetilde{\sigma}_m^2(s)$. In the case of the $R^{1/4}$-law, the result is in agreement with B82: $\widetilde{\sigma}_4$ peaks at $\sim 0.08 R_e$. In Fig. 3 (right, solid line) the logarithm of the 1-dimensional velocity dispersion is shown for $2 \le m \le 10$. It is important to note the presence of the central depression, which is a common property of the $R^{1/m}$-law models, as already pointed out by Binney (1980) for the de Vaucouleurs law. This behavior can be explained in the general way described by Binney (1980), by taking into account Eq. (B6). But, in the particular case of the $R^{1/m}$ stellar models a *direct* approach is perhaps useful. From Eq. (B8) we find that the central value of the velocity dispersion vanishes for any $m > 1$: consequently, near the center, the velocity dispersion can only increase outwards. It is also important to note in Fig. 3 that the central *depression* of the velocity dispersion is large for smaller values of m. This important feature can be easily understood: for a fixed (small) radius, the derivative of the velocity dispersion can be found from Eq. (B8), and it turns out to be proportional to $s^{-(1+m)/2m}$, i.e. it is a decreasing function of m. For $r \to 0$, at higher values of m there corresponds a smaller divergence of the velocity dispersion derivative.

The dashed lines in Fig. 3 refers to the dimensionless line–of–sight velocity dispersion $\widetilde{\sigma}_{Pm}(\eta)$: of course, the dimensional factor of normalization is the same as that for the spatial velocity dispersion, while the dimensionless function is given by

$$\widetilde{\sigma}_{Pm}^2(\eta) = -2\exp(b\eta^{1/m}) \int_\eta^\infty \widetilde{\nu}_m(s) \frac{d\widetilde{\psi}_m(s)}{ds} \sqrt{s^2 - \eta^2} ds. \tag{14}$$

This formula is obtained from the general definition of $\widetilde{\sigma}_{Pm}$ given, e.g., in BT87 (Eq. (4.57), p. 205), integrating by parts and using the Jeans equation. In this way we can eliminate the divergence of the argument of the integral, and obtain a best suited formula for numerical computation. The main feature of the line–of–sight velocity dispersion is the nearly total absence of the previously cited central depression: in any case, from the observational point of view, it is important to note that σ_{Pm} follows closely σ_m, for $0.1 \lesssim \eta \lesssim 10$.

Another useful parameter in the description of a spherical system is the *gravitational radius* r_G, defined by the relation $|W| = GM^2/r_G$ (BT87), where $|W|$ and M are the potential energy and the total mass of the system respectively. Referring to an $R^{1/m}$-law galaxy, if the dimensionless gravitational radius $s_G = r_G/R_e$ is known, from the virial theorem it is possible to derive the total mass of the system as $M_m = R_e < \sigma_{obs}^2 > s_G/G$, where $< \sigma_{obs}^2 >$ is the luminosity weighted velocity dispersion over the whole object (see, e.g. Djorgovski and Davis 1987, Eq. (4)). It is interesting to note that the parameter s_G is weakly sensitive to variations of m. In fact, it is bounded between 3.24 for $m = 2$ and 1.78 for $m = 10$. For the de Vaucouleurs models the value obtained for s_G is in agreement with that given by Poveda (1958).

Finally, for the classification of the models, it is interesting to study the *concentration parameter* s_h/s_G (Spitzer 1969): the higher is the value of this parameter, the higher is the model concentration. As one can see from Tab. 2, s_h/s_G is an increasing function of m, which takes the value 0.41 for $m = 2$, and 0.76 for $m = 10$. For the particular case of the $R^{1/4}$-law, the concentration is ~ 0.45, in agreement with the value given by Stiavelli and Bertin (1985).

3. The distributions DF_m and N_m for the $R^{1/m}$-law models

3.1. $DF_m(\mathscr{E})$

For a general collisionless system the distribution function DF, which can be interpreted as the phase space *density*, is the solution of the collisionless Boltzmann equation, and depends on the phase space variables and time. It is related to the density ρ of the system by the equation:

$$\rho(\boldsymbol{x}, t) = \int DF(\boldsymbol{x}, \boldsymbol{v}, t) d^3\boldsymbol{v}, \tag{15}$$

where the integral is extended to the whole velocity space. A requirement for any physically admissible DF is clearly its non-negativity over the accessible phase-space of the system. If the system is stationary, the DF depends on the phase space co-ordinates *only* through the isolating integrals of motion (*Jeans'*

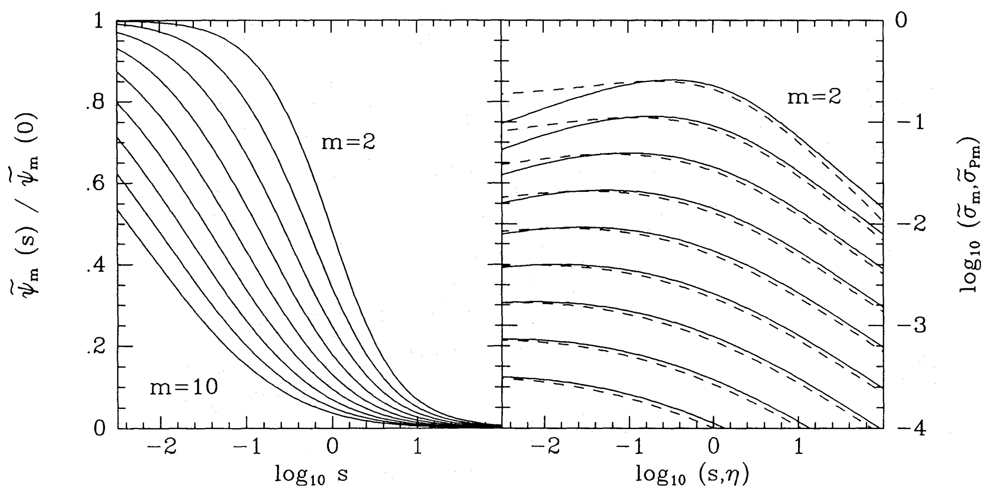

Fig. 3. Left panel: the normalized relative potentials, $\widetilde{\psi}_m(s)/\widetilde{\psi}_m(0)$, for $2 \leq m \leq 10$. Right panel: the one–dimensional velocity dispersions, $\log_{10} \widetilde{\sigma}_m(s)$ for $-2.5 \leq \log_{10} s \leq 2$ (solid lines), and the line–of–sight velocity dispersions, $\log_{10} \widetilde{\sigma}_{Pm}(\eta)$ (dashed lines), for $-2.5 \leq log_{10}\eta \leq 2$

Theorem, Chandrasekhar 1942), and moreover if the system is spherical and its velocity dispersion tensor is isotropic, then its DF depends on the binding energy for unit mass only. Generally, the negative value of the binding energy, the *relative binding energy* $\mathscr{E}$ is used. Thus, indicating with $\psi(r)$ the absolute value of the potential of the system, the positivity condition for the DF is that $DF(\mathscr{E}) \geq 0$ for $0 < \mathscr{E} \leq \psi(0)$ (bounded stars) and $DF(\mathscr{E}) = 0$ for $\mathscr{E} \leq 0$ (unbounded stars).

Using the Eddington technique (Eddington 1916) we can obtain the self-consistently generated $DF(\mathscr{E})$ of a system described by a density-potential pair (ρ, ψ) in the following way:

$$DF(\mathscr{E}) = \frac{1}{\sqrt{8}\pi^2} \int_0^{\mathscr{E}} \frac{d^2\rho}{d\psi^2} \frac{d\psi}{\sqrt{\mathscr{E} - \psi}} + \left(\frac{1}{\sqrt{\mathscr{E}}} \frac{d\rho}{d\psi} \right)_{|\psi=0}. \quad (16)$$

The DF obtained in this way is not automatically physically consistent, in the sense that the Eddington inversion does not guarantee its positivity. If for some positive value of $\mathscr{E}$ the DF is negative, then the assumption of isotropic velocity dispersion is inconsistent with the assumed density profile. For the $R^{1/m}$-law models, the second term in the r.h.s. of Eq. (16) vanishes. In fact, the derivative $\left(d\rho/d\psi\right)_{|\psi=0} = \lim_{r\to\infty} \left(d\rho/dr\right) \left(dr/d\psi\right) \sim O\left(s^{3/2m}\right) \times \exp(-bs^{1/m}) \longrightarrow 0$ for any positive value of m. Thus, defining $\mathscr{E} = G(M/L)I_oR_e \times \widetilde{\mathscr{E}}_m\widetilde{\psi}_m(0)$, where $\widetilde{\mathscr{E}}_m = \mathscr{E}/\psi_m(0)$, we have that $DF_m(\mathscr{E}) = [G^3(M/L)I_oR_e^5]^{-1/2} \times \widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$. The dimensionless term is given by:

$$\widetilde{DF}_m(\widetilde{\mathscr{E}}_m) = \frac{1}{\pi^2\sqrt{8\widetilde{\psi}_m(0)}} \int_{s(\widetilde{\mathscr{E}}_m)}^{\infty} \frac{\Delta_m(x)}{\sqrt{\widetilde{\mathscr{E}}_m - \widetilde{\psi}_m(x)/\widetilde{\psi}_m(0)}} dx, \quad (17)$$

and (B82),

$$\Delta_m(x) = \frac{x^2}{\widetilde{L}_m(x)} \left[\frac{d^2\widetilde{v}_m(x)}{dx^2} + \frac{d\widetilde{v}_m(x)}{dx} \left(\frac{2}{x} - \frac{4\pi x^2 \widetilde{v}_m(x)}{\widetilde{L}_m(x)} \right) \right]. \quad (18)$$

In Eq. (17), the integration variable is transformed from potential to radius $s(\widetilde{\mathscr{E}}_m)$, i.e. the maximum radius reachable by a star of binding energy $\widetilde{\mathscr{E}}_m$. At small radii the DF_m diverges for any m, as stated by Eq. (B6) and showed in Fig. 4 (left), where the family of $\log_{10} \widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ vs. $\widetilde{\mathscr{E}}_m$ is reported. In any case, the central divergence of the DF_m is not a problem: in fact, although the central phase space density of the models diverges, the mass corresponding to this energy does not.

The main result of the computation is the positivity of the $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ for any model, i.e., the $R^{1/m}$-law stellar systems *can be supported by an isotropic velocity dispersion tensor*. Moreover, again from Fig. (4), one can see that the derivative of DF_m with respect to $\mathscr{E}$ is positive for any model: this implies that the $R^{1/m}$ models are stable to radial and nonradial perturbations (see, e.g. BT 87, pp. 306–307).

3.2. $N_m(\mathscr{E})$

For the study of a dynamical system it is also useful to know the differential energy distribution, $dM/d\mathscr{E} = DF(\mathscr{E}) \times G(\mathscr{E})$, where the factor $G(\mathscr{E})d\mathscr{E}$ is the *volume* element of the phase space. For the $R^{1/m}$ models one can express the volume factor as $G_m(\mathscr{E}) = [G(M/L)I_oR_e^7]^{1/2} \times \widetilde{G}_m(\widetilde{\mathscr{E}}_m)$, where:

$$\widetilde{G}_m(\widetilde{\mathscr{E}}_m) = 16\sqrt{2\widetilde{\psi}_m(0)}\pi^2 \int_0^{s(\widetilde{\mathscr{E}}_m)} x^2 \sqrt{\widetilde{\psi}_m(x)/\widetilde{\psi}_m(0) - \widetilde{\mathscr{E}}_m} dx. \quad (19)$$

In Fig. 4 (left), the family of curves $\log_{10} \widetilde{G}_m(\widetilde{\mathscr{E}}_m)$ vs. $\widetilde{\mathscr{E}}_m$ for $2 \leq m \leq 10$ is shown. One can note the central convergence at zero of these functions, as stated by Eq. (B2). I will use the *normalized differential energy distribution*, i.e. the $dM/d\mathscr{E}$ function normalized to total mass M_m, and with $\mathscr{E}$ normalized to the central value of the potential, $\psi_m(0)$. I will indicate this dimensionless function with $N_m(\widetilde{\mathscr{E}}_m)$, so that

$$N_m(\widetilde{\mathscr{E}}_m) = \widetilde{DF}_m(\widetilde{\mathscr{E}}_m) \times \widetilde{G}_m(\widetilde{\mathscr{E}}_m) \times \widetilde{\psi}_m(0)/\widetilde{L}_m. \quad (20)$$

By definition, the integral of $N_m(\widetilde{\mathscr{E}}_m)$ over all the energy range $0 \leq \widetilde{\mathscr{E}}_m \leq 1$ must be 1, and this property can also be used to test the accuracy of the computation. In Fig. 4 (right), the family of $\log_{10} N_m(\widetilde{\mathscr{E}}_m)$ vs. $\widetilde{\mathscr{E}}_m$ is shown. As for the $\widetilde{G}_m(\widetilde{\mathscr{E}}_m)$, the central value for $N_m(\widetilde{\mathscr{E}}_m)$ is zero, as shown by Eq. (B7).

It is evident that the principal characteristic of the $N_m(\widetilde{\mathscr{E}}_m)$ is its exponential behavior over a substantial range of energy. This range and its amplitude can be quantitatively defined by

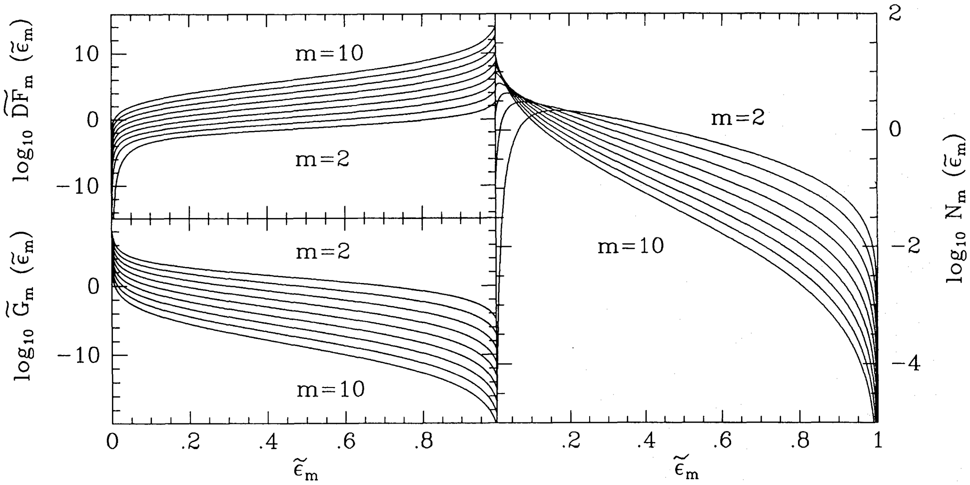

Fig. 4. Left panel: the distribution functions, $\log_{10}\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ (top), and the $\log_{10}\widetilde{G}_m(\widetilde{\mathscr{E}}_m)$ (down) functions. Right panel: the normalized differential energy distributions, $\log_{10}N_m(\widetilde{\mathscr{E}}_m)$ for $2 \le m \le 10$. In the abscissae scale $\widetilde{\mathscr{E}}_m = \mathscr{E}/\psi_m(0)$

fitting the exponential zone using a standard least square method and a parametric Boltzmann formula $N_B(\mathscr{E}) = N_o exp(-\beta\widetilde{\mathscr{E}}_m)$. In the following, I will call *Boltzmannian zone* the widest energy range for which the correlation coefficient of the fit is higher than 0.999, and $\Delta\widetilde{\mathscr{E}}_m$ its amplitude. It turns out that $\Delta\widetilde{\mathscr{E}}_m \simeq 0.5$, nearly *independent* of the parameter m. This shows that the large extension in energy of the Boltzmannian zone is not a special property of the de Vaucouleurs model, but is more a measure of the concentration of the model, as pointed out by Bertin and Stiavelli (1989). As far as the parameter β is concerned, one can note from Tab. 2 that its value is an increasing function of the model concentration, as already pointed out by Stiavelli and Bertin (1985).

Taking into account the previous result, it is natural to ask whether there exists some physically interesting quantity in the family of $R^{1/m}$ stellar systems which is characteristic of the de Vaucouleurs law. In this context I define $\Delta M_m/M_m$ as the mass fraction of the models corresponding to the Boltzmannian zone of the function $N_m(\widetilde{\mathscr{E}}_m)$, i.e. the integral of $N_m(\widetilde{\mathscr{E}}_m)$ over $\Delta\widetilde{\mathscr{E}}_m$. The values of this function are reported in Tab. 2. The main result is that this function has a different behavior for $2 \le m \le 4$, for which it assumes values relatively higher than those for $4 \le m \le 10$. Moreover, $\Delta M_m/M_m$ is maximized by $m = 4$, i.e. in the case of the de Vaucouleurs law.

4. Conclusions

The results of this work are the following:

1) The distribution function $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ is nowhere negative in the phase space, for spherical, isotropic $R^{1/m}$ stellar systems with $2 \le m \le 10$, i.e. these systems are physically admissible. Moreover, $d\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)/d\widetilde{\mathscr{E}}_m > 0$ for $0 < \widetilde{\mathscr{E}}_m < 1$, and this shows that these systems are stable to both radial and nonradial perturbations.

2) The exponential behavior of the differential energy distribution $N_m(\widetilde{\mathscr{E}}_m)$ is a *common* property of all the $R^{1/m}$-law models. Thus, it is *really* an effect of the central concentration.

3) The special property shown by the $R^{1/4}$-law is that the mass fraction corresponding to the Boltzmannian zone is maximized. This is due to the special behavior of the functions $N_m(\widetilde{\mathscr{E}}_m)$ for $r \to \infty$. In fact, while the fraction $\Delta\widetilde{\mathscr{E}}_m$ of the binding energy where the function $N_m(\widetilde{\mathscr{E}}_m)$ is exponential is nearly independent of the value of m, its *position* depends on m. For $10 \ge m \ge 4$, this range moves towards low energies; in contrast, for $4 \ge m \ge 2$ it moves towards high energies. In other words, models with lower concentration than the de Vaucouleurs law are depressed in the external part of the $N_m(\widetilde{\mathscr{E}}_m)$ with respect to a pure Boltzmann law, and models with higher concentration have too high a differential energy distribution: generally, in a stellar system, most of the mass is bound at higher energies, i.e. near the tidal radius, and then the behavior of the $\Delta M_m/M_m$ for the $R^{1/4}$-law model is easily understood.

Acknowledgements. I wish to thank G. Bertin for valuable discussions and for much advice, P. Londrillo for useful comments, and the referee J. Binney for clarifying suggestions. I am also indebted to S. Pellegrini who kindly read versions of the manuscript. This work has been partially supported by the Italian Ministry of Research (MURST).

Appendix A

In this Appendix I derive the asymptotic expansions for $\alpha \to 0$ and $\alpha \to \infty$ of the functions $\mathscr{C}^\beta_m(\alpha)$, and discuss their connection with the derivatives of the luminosity density $\widetilde{\nu}_m$ [see Eq. (7)]. The function $\mathscr{C}^\beta_m(\alpha)$ is given in integral form as:

$$\mathscr{C}^\beta_m(\alpha) = \lim_{X\to\infty}\int_1^X \frac{t^\beta \exp(-\alpha t)dt}{\sqrt{t^{2m}-1}}, \qquad (A1)$$

where β is a real constant, $m > 0$, and $\alpha \ge 0$; we recall that $\alpha = bs^{1/m}$ is the reduced radial coordinate. The derivatives of $\mathscr{C}^\beta_m(\alpha)$ with respect to α, which enter in the expression of $\Delta_m(x)$ [see Eq. (18)], follow the relation:

$$\frac{\partial^n\mathscr{C}^\beta_m(\alpha)}{\partial\alpha^n} = (-1)^n\mathscr{C}^{\beta+n}_m(\alpha). \qquad (A2)$$

The argument in the integral of the $\mathscr{C}^\beta_m(\alpha)$ function is *singular* at $t = 1$, but this singularity is independent of the value of the parameters β and α, and it is integrable for any value of m. The integral in Eq. (A1) diverges at ∞ *only* in the case of $\alpha = 0$ *and* $m - \beta \leq 1$.

For large values of α, we have for any value of β and m:

$$\mathscr{C}_m^\beta(\alpha) \sim \sqrt{\frac{\pi}{2m}} \frac{\exp(-\alpha)}{\sqrt{\alpha}}. \qquad (A3)$$

In fact, for $\alpha \to \infty$, the double limit in Eq. (A1) is zero independently of the order. Thus, changing the variable of integration in Eq. (A1) $t = x/\alpha$, inverting the double limit, and expanding in power series the argument in the integral, we obtain the proof of Eq. (A3). Then, for any value of m, the asymptotic behavior of the luminosity density for $\alpha \to \infty$ is shown in Eq. (8). It is also interesting to note that the first term of the asymptotic expansion of $\mathscr{C}_m^\beta(\alpha)$ at infinity is independent of β.

For $\alpha \to 0$, three different results are obtained for $m - \beta > 1$, $m - \beta = 1$, and $m - \beta < 1$. In the case $m - \beta > 1$, the integral (A1) converges at $\alpha = 0$: changing the variable of integration $x = t - 1$, we have

$$\mathscr{C}_m^\beta(\alpha) \sim \frac{B[1/2, (m-1)/2m]}{2m} \exp(-\alpha). \qquad (A4)$$

For $\beta = 0$ we have that, for $m > 1$,

$$\widetilde{v}_m(\alpha) \sim \frac{B[1/2, (m-1)/2m]}{2\pi m b^{-m}} \exp(-\alpha)\alpha^{1-m}. \qquad (A5)$$

Thus, the luminosity density for $\alpha \to 0$ diverges as shown in Eq. (9).

In the case of $m - \beta = 1$, setting $\beta = 0$ in Eq. (A1) and changing the variable of integration $t = \cosh(x)$, we have $\widetilde{v}_1(\alpha) = b(1)K_0(\alpha)/\pi$ where K_0 is the zeroth-order modified Bessel function of the third kind (EMOT53, vol. II, p. 5-9), and then

$$\widetilde{v}_1(\alpha) \sim \frac{b(1)}{\pi} \ln\left(\frac{2}{\alpha}\right). \qquad (A6)$$

Finally, in the case $m - \beta < 1$, if one changes $x = \alpha t$ in Eq. (A1), for small radii we have

$$\mathscr{C}_m^\beta(\alpha) \sim \Gamma(1 + \beta - m)\alpha^{m-\beta-1}, \qquad (A7)$$

and

$$\widetilde{v}_m(\alpha) \sim \frac{b^m\Gamma(1-m)}{\pi}. \qquad (A8)$$

Appendix B

In this Appendix I record the asymptotic behavior for $s \to 0$ and $m > 1$ of the functions $\widetilde{G}_m(\widetilde{\mathscr{E}}_m)$, $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$, and $N_m(\widetilde{\mathscr{E}}_m)$. The first therm of the asymptotic expansion of the velocity dispersion for $s \to 0$ is also given.

The density for $s \to 0$ is given in Eq. (9), which can be written as $\rho_m(s) \sim c_0(m) \times s^\lambda$, where $\lambda = (1-m)/m$ and $-1 < \lambda < 0$. Therefore, the mass inside s is given by $M_m(s) \sim c_1(m)s^{\lambda+3}$, where $c_0(m)$ and $c_1(m)$ are positive constants. Then, the potential difference required in Eqs. [(18),(19)] is given by

$$\widetilde{\psi}_m(x) - \widetilde{\psi}_m(s) \sim c_2(m) \left(s^{\lambda+2} - x^{\lambda+2}\right); \qquad c_2(m) > 0. \qquad (B1)$$

Inserting Eq. (B1) in Eq. (19), and normalizing the variable of integration to $s(\widetilde{\mathscr{E}}_m)$, we have:

$$\widetilde{G}_m(\widetilde{\mathscr{E}}_m) \sim c_3(m) \times s(\widetilde{\mathscr{E}}_m)^{(1+7m)/2m}; \qquad c_3(m) > 0. \qquad (B2)$$

So, we have the property that the asymptotic behavior of $\widetilde{G}_m(\widetilde{\mathscr{E}}_m)$ for $s \to 0$ is nearly independent of m, with the exponent ranging from 3.5 for $m \to \infty$ to 4 for $m \to 1$.

The situation is more complicated for the distribution function $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$. From Eq. (17) we should note that only the *sign* of the function $\Delta_m(x)$ can determine the positivity of the distribution function. It is also straightforward to show that the asymptotic behavior of this function for $x \to \infty$ is *exponentially* decreasing for *any* value of m (see Eqs. (8),(18)). Thus, if for $s \to 0$,the function $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ diverges, its sign and its asymptotic behavior are given by the integral of the asymptotic expansion of the integrand close to the center. For $m > 1$,

$$\Delta_m(x) \sim \frac{c_4(m)}{x^3}; \qquad c_4(m) > 0. \qquad (B3)$$

As far as the square root in the denominator of Eq. (17) is concerned, Eq. (B1) must be used. Defining

$$I_{\mu,\nu}(A, s) = \int_s^A \frac{dx}{x^\mu\sqrt{x^{2\nu} - s^{2\nu}}}; \qquad (\nu \neq 0), \qquad (B4)$$

for $s \to 0$, we have:

$$I_{\mu,\nu}(A, s) \sim \begin{cases} O(s^{1-\mu-\nu}); & \text{if } \mu + \nu > 1; \\ O[-\ln(s)]; & \text{if } \mu + \nu = 1; \\ O(1); & \text{if } \mu + \nu < 1. \end{cases} \qquad (B5)$$

Then, in our case, from Eqs. [(B1),(B3)], we have $\mu = 3$ and $\nu = (1+m)/2m$: these values make the integral in Eq. (17) diverge. Thus, the $\widetilde{DF}_m(\widetilde{\mathscr{E}}_m)$ diverges at the center as:

$$\widetilde{DF}_m(\widetilde{\mathscr{E}}_m) \sim c_5(m) \times s(\widetilde{\mathscr{E}}_m)^{-(1+5m)/2m}; \qquad c_5(m) > 0. \qquad (B6)$$

The exponent of Eq. (B6) spans the range $]-3, -2.5[$ for $1 < m < \infty$. Finally, the central behavior of the differential energy distribution is given by:

$$N_m(\widetilde{\mathscr{E}}_m) \sim c_6(m) \times s(\widetilde{\mathscr{E}}_m); \qquad c_6(m) > 0, \qquad (B7)$$

i.e. it is independent of m.

The velocity dispersion is zero for $s \to 0$ and $m > 1$. In fact, if we consider Eq. (13), the integral at the r.h.s. converges to some positive constant. Thus, the l.h.s. also converges, and the first term of the asymptotic expansion for the velocity dispersion is:

$$\widetilde{\sigma}_m(s) \sim c_7(m) \times s^{-\lambda/2}; \qquad c_7(m) > 0. \qquad (B8)$$

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