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ON A FAMILY OF CURIOUS INTEGRALS SUGGESTED BY STELLAR DYNAMICS

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Abstract

While investigating the properties of a galaxy model used in Stellar Dynamics, a curious integral identity was discovered. For a special value of a parameter, the identity reduces to a definite integral with a very simple symbolic value; but, quite surprisingly, all the consulted tables of integrals, and computer algebra systems, do not seem aware of this result. Here I show that this result is a special case ($n = 0$ and $z = 1$) of the following identity (established by elementary methods):

$$I_n(z) \equiv \int_0^1 \frac{\mathrm{K}(k)k}{(z+k^2)^{n+3/2}}dk = \frac{(-2)^n}{(2n+1)!!}\frac{d^n}{dz^n}\frac{\mathrm{ArcCot}\sqrt{z}}{\sqrt{z(z+1)}}, \quad z > 0, \tag{1}$$

where $n = 0, 1, 2, 3...$, and $\mathrm{K}(k)$ is the complete elliptic integral of first kind.

1. Introduction

During their investigation of the dynamical properties (in particular, the expression for the self-gravitational energy) of a galaxy model often used in Astronomy, Baes & Ciotti (2019) found the quite unexpected closed form identity

$$\int_0^1 \frac{\mathrm{K}(k)k}{(1+k^{1/m})^{3m}}dk = \frac{\pi m}{4\Gamma(3m)}\mathrm{H}^{2,2}_{3,3}\left[1\Big|\begin{matrix}(1-3m,2m) & (0,1) & (0,1)\\ (0,2m) & (-1/2,1) & (-1/2,1)\end{matrix}\right], \tag{2}$$

where $m > 0$ is a real number,

$$\mathrm{K}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \mathrm{F}\left(\frac{\pi}{2}, k\right), \tag{3}$$

is the standard complete elliptic integral of first kind in Legendre form[1], and finally $\mathrm{H}^{m,n}_{p,q}$ is the *Fox*-H *function* (e.g., see Mathai et al. 2009, see also Prudnikov 1990). Identity (2) was established by evaluating the self-gravitational energy of the model following two

[1] Note that in Mathematica, $K(k) = \mathrm{EllipticK}[k^2]$.

different paths: the one leading to the expression in terms of $\mathrm{H}^{m,n}_{p,q}$ is based on successive integrations starting from an Abel inversion; the other, involving the elliptic integral, is obtained by repeated exchange of the order of integration in a multidimensional integral (for details see Baes and Ciotti 2019).

For $m = 1/2$, the $\mathrm{H}^{m,n}_{p,q}$ function reduces to a *Meijer*-$\mathrm{G}^{m,n}_{p,q}$ *function* (e.g., see Gradshteyn and Ryzhik 2007), with an already known value expressed in terms of known constants, and identity (2) was found to reduce to

$$I_0(1) \equiv \int_0^1 \frac{\mathrm{K}(k)k}{(1+k^2)^{3/2}} dk = \frac{\pi}{4\sqrt{2}}; \tag{4}$$

for reasons that will be clear soon I call $I_0(1)$ the integral above. Surprisingly, the simple-looking identity (4) is not found in the tables of integrals of common use or specialized works on elliptic integrals, (e.g., Erdélyi et al. 1953, Gradshteyn and Ryzhik 2007, Prudnikov et al. 1990, see also Baes and Ciotti 2019 and references therein), and neither the latest releases of Mathematica and Maple seems to be able to recover the result.

Prompted by the curious identity (4), I found that not only it can be established by elementary methods, but it is generalized to identity (1). For example, the first few integrals in the family are

$$I_0(1) = \frac{\pi}{4\sqrt{2}}, \quad I_1(1) = \frac{1}{6\sqrt{2}} + \frac{\pi}{8\sqrt{2}}, \quad I_2(1) = \frac{1}{6\sqrt{2}} + \frac{19\pi}{240\sqrt{2}}, \quad I_3(1) = \frac{121}{840\sqrt{2}} + \frac{9\pi}{160\sqrt{2}}, \tag{5}$$

so that for arbitrary integers $n \geq 0$ and $m \geq 0$, the numbers $I_n(1)$ and $I_m(1)$ are related as $\sqrt{2}I_n(1) + P(n,m)\sqrt{2}I_m(1) + Q(n,m) = 0$, with $P(n,m)$ and $Q(n,m)$ rational numbers, as can be easily proved from the Leibniz product rule applied to the r.h.s. of identity (1).

Given the functional form of $I_n(z)$, it is not surprising that an endless number of identities similar to those in eq. (5) can be obtained with (almost) no efforts by using special algebraic values of z such as $z = 1/3$, $z = 3$, or $z = \mathrm{Cot}^2(\pi/10) = 5 + 2\sqrt{5}$, $z = \mathrm{Cot}^2(\pi/12) = 7 + 4\sqrt{3}$, and so on. For example, we obtain

$$I_0(3) = \frac{\pi}{12\sqrt{3}}, \quad I_1(3) = \frac{1}{72} + \frac{7\pi}{432\sqrt{3}}, \quad I_2(3) = \frac{1}{180} + \frac{11\pi}{2880\sqrt{2}}, \quad \text{etc} \tag{6}$$

$$I_0(1/3) = \frac{\pi}{2}, \quad I_1(1/3) = \frac{3\sqrt{3}}{8} + \frac{5\pi}{8}, \quad I_2(1/3) = \frac{9\sqrt{3}}{10} + \frac{177\pi}{160}, \quad \text{etc} \tag{6'}$$

or the even more strange-looking

$$I_0(5+2\sqrt{5}) = \frac{\pi}{10\sqrt{50+22\sqrt{5}}}, \quad I_0(7+4\sqrt{3}) = \frac{\pi}{24\sqrt{26+15\sqrt{3}}}, \quad \text{etc.} \tag{6''}$$

2. A proof of identity (1)

Identity (1) is proved by differentiation under sign of integration, first recognizing by induction that

$$I_n(z) = \frac{(-2)^n}{(2n+1)!!} \frac{d^n I_0(z)}{dz^n}, \qquad n = 0, 1, 2, 3.... \tag{7}$$

Therefore the problem reduces to the evaluation of $I_0(z)$. This is done by inversion of order of integration:

$$I_0(z) = \int_0^1 \frac{dt}{\sqrt{1-t^2}} \int_0^1 \frac{k\ dk}{(z+k^2)^{3/2}\sqrt{1-k^2t^2}}. \tag{8}$$

The inner integral is elementary by changing variables as $\sqrt{1-k^2t^2} = x$ and then as $x/\sqrt{1+zt^2} = y$, so that

$$\int_0^1 \frac{k\ dk}{(z+k^2)^{3/2}\sqrt{1-k^2t^2}} = \frac{1}{\sqrt{z}(1+zt^2)} - \frac{\sqrt{1-t^2}}{\sqrt{1+z}(1+zt^2)}. \tag{9}$$

Inserting eq. (9) in eq. (8) leads to compute two integrals. The second integral is trivial

$$-\frac{1}{\sqrt{1+z}} \int_0^1 \frac{dt}{1+zt^2} = -\frac{\mathrm{ArcTan}\sqrt{z}}{\sqrt{z(z+1)}}. \tag{10}$$

The first integral is (slightly) more tricky. First the standard trigonometric substitution $t = \sin x$ is performed, followed by a transformation obtained from $1 = \cos^2 x + \sin^2 x$ as:

$$\frac{1}{\sqrt{z}} \int_0^1 \frac{dt}{(1+zt^2)\sqrt{1-t^2}} = \frac{1}{\sqrt{z}} \int_0^{\pi/2} \frac{dx}{1+z\sin^2 x} = \frac{1}{\sqrt{z}} \int_0^{\pi/2} \frac{dx}{\cos^2 x[1+(1+z)\tan^2 x]}. \tag{11}$$

The last integration is performed with the change of variable $\tan x = y$, so that finally

$$\frac{1}{\sqrt{z}} \int_0^1 \frac{dt}{(1+zt^2)\sqrt{1-t^2}} = \frac{\pi}{2\sqrt{z(z+1)}}. \tag{12}$$

Adding eq. (10) and (12) we obtain the desired result.

3. Conclusion

An elementary derivation is presented for the closed-form expression of a family of definite integrals involving the complete elliptic integral of first kind. The original problem was motivated by an investigation in the field of Stellar Dynamics, where some unexpected identity was established. The integrals, albeit simple looking (and perhaps already evaluated in the literature), seem to be missing in the most common tables of integrals, and also well known computer algebra systems appear unable to evaluate them. From the astrophysical point of view these integrals are just as a mathematical curiosity, but it would be interesting to know something more about the properties of the rational/radicals numerical terms appearing in eqs. (5)-(6”).

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4. References

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