Astro. Lett. and Communications, Vol. 40, pp. 85–91
Reprints available directly from the publisher
Photocopying permitted by license only

INVERSION OF THE ABEL EQUATION FOR TOROIDAL DENSITY DISTRIBUTIONS

L. CIOTTI*

Osservatorio Astronomico di Bologna, via Ranzani 1, 40127 Bologna, Italy

(Received 1 January 1999; In final form 10 March 1999)

In this paper I present three new results of astronomical interest concerning the theory of Abel inversion. (1) I show that in the case of a spatial emissivity that is constant on toroidal surfaces and projected along the symmetry axis perpendicular to the torus' equatorial plane, it is possible to invert the projection integral. From the surface (*i.e.*, projected) brightness profile one then formally recovers the original spatial distribution as a function of the toroidal radius. (2) By applying the above-described inversion formula, I show that if the projected profile is described by a truncated off-center gaussian, the functional form of the related spatial emissivity is very simple and – most important – nowhere negative for any value of the gaussian parameters, a property which is not guaranteed – in general – by Abel inversion. (3) Finally, I show how a generic multimodal centrally symmetric brightness distribution can be deprojected using a sum of truncated off-center gaussians, recovering the spatial emissivity as a sum of nowhere negative toroidal distributions.

Keywords: Methods: analytical; methods: numerical; methods: data analysis

1. INTRODUCTION

A common problem in astronomy is the deprojection of a given surface brightness distribution. Unfortunately this problem is degenerate, *i.e.*, different spatial emissivities can originate the same surface (projected) brightness distribution. As a consequence, any inversion procedure is invariably based on a (more or less) arbitrary choice of the underlying geometry of the spatial emissivity. A help to this unpleasant situation comes from two guide rules: in choosing the

*e-mail: ciotti@astbo3.bo.astro.it

geometry of the spatial emissivity one uses symmetry properties (if any) of the brightness distribution, and after deprojection discards the assumed geometry if this produces a somewhere negative spatial emissivity.

One of the simplest cases is given by a surface brightness characterized by central symmetry, *i.e.*, described by a function $I(R)$ of the projected radius R. The natural assumption is a spherically symmetric spatial emissivity, $\mathcal{E} = \mathcal{E}(r) \geq 0$, where r is the spherical radius. It is a well known result that in this case the projection operator and its deprojection are given by an Abel integral equation:

$$I(R) = 2 \int_R^{r_t} \frac{\mathcal{E}(r) r dr}{\sqrt{r^2 - R^2}}, \tag{1}$$

and

$$\mathcal{E}(r) = -\frac{1}{\pi} \left[\int_r^{r_t} \frac{dI(R)}{dR} \frac{dR}{\sqrt{R^2 - r^2}} - \frac{I(r_t)}{\sqrt{r_t^2 - r^2}} \right], \tag{2}$$

where r_t is the truncation radius, *i.e.*, $I(R) = 0$ for $r > r_t$, and for untruncated distributions $r_t \to \infty$. It is important to note that if $I(r_t) > 0$ the recovered emissivity is (weakly) divergent at the edge of the sphere, due to the second term on the r.h.s. of Eq. (2). A fundamental problem of Abel inversion is that the positivity of the recovered distribution is not guaranteed. As an example of this fact, one can consider $I(R)$ to be a gaussian or an off-center gaussian: while in the first case the deprojected emissivity – being still a gaussian – is everywhere positive (see, for applications, Bendinelli, Ciotti and Parmeggiani, 1993, and references therein), in the second case it can be easily shown that $\mathcal{E}(r)$ diverges negatively for $r \to 0$.

What other symmetries for $\mathcal{E}$ are compatible with a surface brightness $I = I(R)$ and are still of astrophysical interest? The natural generalization of the spherical symmetry is the cylindrical one. Many astronomical objects are characterized by this geometry: lenticular and spiral galaxies, accretion disks, plasma tori, planetary rings, some planetary nebula *etc.* In this paper we are interested in particular in the problem of deprojecting a $I(R)$ which presents some off-center maxima, and so the natural assumption about the emissivity distribution $\mathcal{E}$ is the toroidal one.

In Section 2, I show that in the case of a toroidal emissivity distribution projected along its symmetry axis it is possible to generalize the inversion formula (2). Then I show that in the particular case of a projected brightness described by a truncated off-center gaussian, $\mathcal{E}$ has an extremely simple functional form, and at variance with the spherically symmetric case, *it is nowhere negative for any choice of the gaussian parameters.* In Section 3, I propose the use of the found physically admissible I-$\mathcal{E}$ pair to obtain a (finite) series expansion of the emissivity for a *generic* centrally symmetric surface brightness profile. In a following paper (Bendinelli *et al.*, 1999), an application of this method to the Planetary Nebula (PN) A 13 is described. This PN is just one case among many others of astrophysical interest to which the method produces useful results on the object's structure.

2. THE INVERSION FORMULA FOR TOROIDAL SYMMETRY

Let us start by obtaining the analogous formula of Eq. (1) for the projection along the symmetry axis of an emissivity distribution stratified on toroidal surfaces. The natural coordinates are the cylindrical, (R, φ, z), where $z = 0$ is the torus equatorial plane. The relation between cylindrical and toroidal coordinates (r, φ, ϑ) is given by

$$R = R_o + r \sin\vartheta, \quad \varphi = \varphi, \quad z = r\cos\vartheta, \tag{3}$$

(see Fig. 1).

We assume independence[1] of $\mathcal{E}$ from ϑ and φ, and so each isoemissivity surface is labeled by its toroidal radius $r = \sqrt{z^2 + (R - R_o)^2}$ around the circle of radius R_o placed in the equatorial plane. Moreover, for the sake of generality let us assume that the emissivity is non-zero only for $0 \leq r \leq r_t \leq R_o$. For $r_t = R_o$ we have a so-called *full torus*. With the previous assumptions, $I(R) \geq 0$ for $|R - R_o| \leq r_t$ and $I(R) = 0$ outside. From very simple geometric

[1]The request of independence of $\mathcal{E}$ from φ is unnecessary, and the following discussion can be easily generalized to distributions $I = I(R, \varphi)$, obtaining $\mathcal{E} = \mathcal{E}(r, \varphi)$.

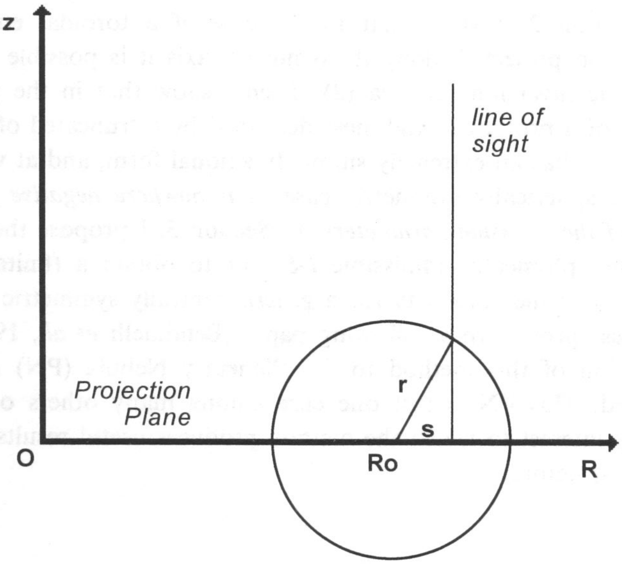

FIGURE 1 The describing parameters of toroidal geometry, and their relations used for the projection and deprojection.

arguments one obtains the analogous formula of Eq. (1):

$$I(R) = 2\int_{|R-R_o|}^{r_t} \frac{\mathcal{E}(r)\,rdr}{\sqrt{r^2 - (R-R_o)^2}}. \tag{4}$$

With respect to the variable R, the l.h.s. of Eq. (4) is a generalized Abel integral, but it is not directly invertible, since $(R-R_o)^2$ is not strictly increasing with R (see Gorenflo and Vessella, 1991, p. 24). Since the function's profile is symmetric with respect to R_o, we can use a branch of I (*i.e.*, we use $R \geq R_o$) and we define a new variable $s := R - R_o \geq 0$ with $I_+(s) := I(R_o + s)$. In this way, the integral is formally identical to the projection operator in Eq. (1) and can be inverted:

$$\mathcal{E}(r) = -\frac{1}{\pi}\left[\int_r^{r_t} \frac{dI_+}{ds}\frac{ds}{\sqrt{s^2 - r^2}} - \frac{I_+(r_t)}{\sqrt{r_t^2 - r^2}}\right]. \tag{5}$$

Thus, we proved that *assuming the surface brightness distribution to be the projection on its equatorial plane of a toroidally stratified emissivity,* *it is formally possible to invert the projection integral, recovering the emissivity as a function of the toroidal radius.*

3. TRUNCATED OFF-CENTER GAUSSIANS

As an application of Eq. (5) we invert a surface brightness profile described by an off-center gaussian truncated at r_t:

$$I(R) = S\left\{ \exp\left[-\frac{(R-R_o)^2}{2\sigma^2}\right] - \exp\left(-\frac{r_t^2}{2\sigma^2}\right)\right\}, \quad |R-R_o| \leq r_t. \quad (6)$$

Note that $I_+(r_t) = 0$, and so the unpleasant divergence at the edge of the torus is avoided. The total luminosity L associated to I is given by its surface integral over the annulus $R_o - r_t \leq R \leq R_o + r_t$:

$$L = SR_o\sigma(2\pi)^{3/2}\left[\mathrm{Erf}\left(\frac{r_t}{\sqrt{2}\sigma}\right) - \frac{r_t\sqrt{2}}{\sigma\sqrt{\pi}}\exp\left(-\frac{r_t^2}{2\sigma^2}\right)\right], \quad (7)$$

where $\mathrm{Erf}(x) = 2/\sqrt{\pi}\int_0^x \exp(-t^2)dt$ is the error function. The formula for the luminosity density given using Eqs. (5), (6) results to be:

$$\mathcal{E}(r) = \frac{S}{\sqrt{2\pi}\sigma}\exp\left(-\frac{r^2}{2\sigma^2}\right)\mathrm{Erf}\left(\sqrt{\frac{r_t^2 - r^2}{2\sigma^2}}\right). \quad (8)$$

The deprojection formula can be verified by inserting Eq. (8) in Eq. (4) and then evaluating the integral. Note that the spatial emissivity given by Eq. (8) is *everywhere positive*, for any choice of the gaussian parameters (S, R_o, r_t, σ). One can then conclude with the following analogy: *off-centered gaussians in toroidal symmetry correspond to centered gaussians in spherical symmetry*.

Having found a nowhere negative I-$\mathcal{E}$ pair that satisfies the Abel inversion for a particular toroidal density distributions, the successive step is the natural extension of a previous work (Bendinelli *et al.*, 1993), where a multigaussian expansion of observed profiles is described, under the assumption of spherical symmetry: here it is sufficient to remember that by using gaussian functions one avoids the computational difficulty of direct Abel numerical inversion, which belongs to the class of unstable inverse problems (Gorenflo and Vessella, 1991; Craig and Brown, 1986).

So one can assume that a given profile $I(R)$ is expanded as a sum of truncated off-center gaussians with different parameters $(S, R_o, r_t, \sigma)_i$ for $i = 1, \ldots, N$. The parameters are easily computable using the Newton–Gauss regularized method which is a powerful iterative non-linear fitting technique (Bendinelli *et al.*, 1987; Bendinelli, 1991). Clearly the parameters are constrained to satisfy the request that the integral over all the projection plane of the fitted brightness distribution must be equal to the original one, *i.e.*, the total luminosity of the system $L = \Sigma_i L_i$, where L_i are given by Eq. (7). The associated spatial emissivity is then approximated by the sum of N distributions as in Eq. (8). The developed technique will be applied to the deprojection of the galactic PN A 13, which is characterized by a well defined ring-shaped morphology (Bendinelli *et al.*, 1999).

4. CONCLUSIONS

The results of this work are the following:

(1) It is shown that it is possible to extend the classical Abel inversion for spherically symmetric systems to toroidal density distributions projected along the torus symmetry axis.
(2) It is also shown that an off-center truncated gaussian function gives rise to a well behaved spatial emissivity, *i.e.*, the emissivity is a very simple function of the toroidal radius. More important, the emissivity is non-negative over all the space for any choice of its parameters, a property not guaranteed by the Abel inversion.
(3) Finally, it is proposed the use of the found I-$\mathcal{E}$ pair for the recovering of the spatial emissivity of any centrally symmetric projected distribution as a sum of toroidal density distribution, after a non-linear parameter fitting.

Acknowledgements

I would like to thank O. Bendinelli, G. Parmeggiani, and L. Stanghellini for useful discussions. This work was partially supported by the contracts ASI-95-RS-152 and ASI-ARS-96-70.

References

Bendinelli, O., Ciotti, L., Parmeggiani, G. and Stanghellini, L. (1999) in preparation.
Bendinelli, O. (1991) *ApJ*, **366**, 599.
Bendinelli, O., Parmeggiani, G., Piccioni, A. and Zavatti, F. (1987) *AJ*, **94**, 1095.
Bendinelli, O., Ciotti, L. and Parmeggiani, G. (1993) *A & A*, **279**, 668.
Craig, I. J. D. and Brown, J. C. (1986) "*Inverse Problems in Astronomy*", Adam Hilger, Bristol.
Gorenflo, R. and Vessella, S. (1991) "*Abel Integral Equations*", Springer-Verlag, Berlin.