

Exact density–potential pairs from the holomorphic Coulomb field

Luca Ciotti^{1★} and Giacomo Giampieri^{2†}

¹*Astronomy Department, University of Bologna, via Ranzani 1, 40127 Bologna, Italy*

²*Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109, USA*

Accepted 2007 January 12. Received 2007 January 8; in original form 2006 December 12

ABSTRACT

We show how the complex-shift method developed by Appell to study the gravitational field of a point mass (and used in electrodynamics by, among others, Newman, Carter, Lynden-Bell, and Kaiser to determine some remarkable properties of the electromagnetic field of rotating charged configurations) can be extended to obtain new and explicit density–potential pairs for self-gravitating systems departing significantly from spherical symmetry. The rotational properties of two axisymmetric baroclinic gaseous configurations derived with the proposed method are illustrated.

Key words: stellar dynamics – celestial mechanics – galaxies: kinematics and dynamics.

1 INTRODUCTION

Potential theory is the cause of major difficulties in astrophysical problems in which gravity is important. In general, to calculate the gravitational potential associated with a given density distribution it is necessary to evaluate a three-dimensional integral. Except for special circumstances in which a solution can be found in terms of elementary functions, this requires resorting to numerical techniques and to sophisticated tools such as expansions in orthogonal functions or integral transforms.

Under spherical symmetry, the density–potential relationship can be reduced to a one-dimensional integral, while for axisymmetric systems it is generally the case that a (usually non-trivial) two-dimensional integral remains. As a result, the majority of the available explicit density–potential pairs refer to spherical symmetry, and only a few axially symmetric pairs are known (e.g. see Binney & Tremaine 1987, hereafter BT). In special cases (in particular when a density–potential pair can be expressed in a suitable parametric form) there exist systematic procedures with which to generate new non-trivial density–potential pairs (see, for example, the case of Miyamoto & Nagai 1975 and the related Satoh 1980 discs; see also Evans & de Zeeuw 1992; de Zeeuw & Carollo 1996). For non-axisymmetric systems the situation is worse. One class of triaxial density distributions for which the potential can be expressed in a tractable integral form is that of the stratified homeoids, such as the Ferrers (1887) distributions (e.g. see Pfenniger 1984; Lanzoni & Ciotti 2003, and references therein) and special cases of the family considered by de Zeeuw & Pfenniger (1988). Additional explicit density–potential pairs are given by the Evans (1994) models and by those constructed with the Kutuzov–Osipkov method (Kutuzov & Osipkov 1980; see also Kutuzov 1998). Another way to construct

in a systematic way explicit density–potential pairs with finite deviations from spherical symmetry is presented in Ciotti & Bertin (2005). This technique is based on an elementary property of the asymptotic expansion of the homeoidal potential quadrature formula for small flattenings,¹ and has recently been applied to the modelling of gaseous haloes in clusters (Lee & Suto 2003, 2004), to the study of the dynamics of elliptical galaxies (Muccione & Ciotti 2003, 2004), and to a possible interpretation of the rotational field of the extraplanar gas in disc galaxies (Barnabé et al. 2005, 2006).

In the context of classical electrodynamics a truly remarkable and seemingly unrelated result was obtained by Newman (1973, see also Newman & Janis 1965; Newman et al. 1965), who considered the case of the electromagnetic field of a point charge displaced on the imaginary axis. Successive analyses by Carter (1968), Lynden-Bell (2000, 2002, 2004a, b, and references therein; see also Teukolsky 1973; Chandrasekhar 1976; Page 1976), and Kaiser (2004, and references therein) revealed the astonishing properties of the resulting holomorphic ‘magic’ field. As pointed out to us by the referee, the complex-shift method was first introduced by Appell (1887, see also Whittaker & Watson 1950) to the case of the gravitational field of a point mass, and subsequently used in general relativity (e.g. see Gleiser & Pullin 1989; Letelier & Oliveira 1987, 1998; D’Afonseca, Letelier & Oliveira 2005, and references therein).

In this paper we show how the complex-shift method can be applied in classical gravitation to obtain exact and explicit density–potential pairs deviating significantly from spherical symmetry; we are not aware that this possibility has been examined in the literature. The paper is organized as follows. In Section 2 we present the idea behind the method, and discuss its application to the case of spherical systems, describing two new self-gravitating and axisymmetric systems obtained from the Plummer and isochrone spheres.

*E-mail: luca.ciotti@unibo.it

†Deceased

¹ Such expansion can be traced back to the treatise on geodesy by Sir H. Jeffreys (1970, and references therein; see also Hunter 1977).

In Section 3 we focus on the rotational fields of the new pairs when interpreted as gaseous systems. Section 4 summarizes the main results obtained, and in the Appendix we examine the behaviour of a singular isothermal sphere under the action of the complex shift.

2 GENERAL CONSIDERATIONS

We start by extending the complexification of a point-charge Coulomb field discussed by Lynden-Bell (2004b) to the gravitational potential $\Phi(\mathbf{x})$ generated by a density distribution $\rho(\mathbf{x})$. From here on, $\mathbf{x} = (x, y, z)$ will indicate the position vector, and $\langle \mathbf{x}, \mathbf{y} \rangle \equiv x_i y_i$ is the standard inner product over the reals (repeated index summation convention implied).

Let us assume that the (nowhere negative) density distribution $\rho(\mathbf{x})$ satisfies the Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho, \quad (1)$$

and that the associated complexified potential Φ_c with shift $i\mathbf{a}$ is defined as

$$\Phi_c(\mathbf{x}) \equiv \Phi(\mathbf{x} - i\mathbf{a}), \quad (2)$$

where $i^2 = -1$ is the imaginary unit and \mathbf{a} is a real vector. The idea behind the proposed method is based on the recognition (1) that the Poisson equation is a linear partial differential equation, and (2) that the complex shift is a linear coordinate transformation. From these two properties, and from equations (1) and (2), it follows that

$$\nabla^2 \Phi_c = 4\pi G \rho_c, \quad (3)$$

where

$$\rho_c(\mathbf{x}) \equiv \rho(\mathbf{x} - i\mathbf{a}). \quad (4)$$

Thus, by separating the real and imaginary parts of Φ_c and ρ_c obtained from the shift of a known real density–potential pair one obtains *two* real density–potential pairs.

It is pertinent to draw attention here to a distinction between electrostatic and gravitational problems: while in the former case a density (charge) distribution with negative and positive regions can be (at least formally) accepted, in the gravitational case the obtained density components have physical meaning only if they do not change sign (see, however, Section 4). It is interesting that a general result concerning the sign of the real and imaginary parts of the shifted density can be obtained by considering the behaviour of the complexified self-gravitational energy and total mass. In fact, from the linearity of the shift, it follows that the volume integral over the entire space,

$$\begin{aligned} W_c &\equiv \frac{1}{2} \int \rho_c \Phi_c d^3 \mathbf{x} \\ &= \frac{1}{2} \int [\Re(\rho_c) \Re(\Phi_c) - \Im(\rho_c) \Im(\Phi_c)] d^3 \mathbf{x}, \end{aligned} \quad (5)$$

coincides with the self-gravitational energy $W = 0.5 \int \rho \Phi d^3 \mathbf{x}$ of the real unshifted seed density. Thus the imaginary part of W_c is zero; that is,

$$\begin{aligned} &\frac{1}{2} \int [\Re(\rho_c) \Im(\Phi_c) + \Im(\rho_c) \Re(\Phi_c)] d^3 \mathbf{x} = \\ &-G \int \int \frac{\Re[\rho_c(\mathbf{x})] \Im[\rho_c(\mathbf{x}')] }{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x} d^3 \mathbf{x}' = 0, \end{aligned} \quad (6)$$

and W_c is the difference of the gravitational energies of the real and the imaginary parts of the shifted density. The vanishing of the

double integral (6) shows that the integrand necessarily changes sign; that is, the complex shift cannot generate *two* physically acceptable densities. Additional information is provided by considering that, again for the above reasons, it is also the case that the total mass of the complexified distribution $M_c = \int \rho_c d^3 \mathbf{x}$ coincides with the total (real) mass of the seed density distribution $M = \int \rho d^3 \mathbf{x}$, so that

$$\int \Im(\rho_c) d^3 \mathbf{x} = 0. \quad (7)$$

Identities (6)–(7) then leave open the question whether at least $\Re(\rho_c)$ can be characterized by a single sign over the whole space. It turns out that it can either change sign or not, depending on the specific seed density and shift vector adopted. In fact, in the following section we show that simple seed densities exist so that $\Re(\rho_c)$ is positive everywhere, while in the Appendix we present a simple case in which $\Re(\rho_c)$ changes sign.

2.1 Spherically symmetric ‘seed’ potential

We now restrict the previous general considerations to a spherically symmetric real potential $\Phi(r)$, where $r \equiv |\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is the spherical radius, and $|\dots|$ is the standard Euclidean norm. After the complex shift the norm must still be interpreted over the reals² (e.g. see Lynden-Bell 2004b), so that

$$|\mathbf{x} - i\mathbf{a}|^2 = r^2 - 2i \langle \mathbf{x}, \mathbf{a} \rangle - a^2, \quad (8)$$

where $a^2 \equiv |\mathbf{a}|^2$. Without loss of generality we assume that $\mathbf{a} = (0, 0, a)$, and the shifted radial coordinate becomes

$$r_c^2 = r^2 - 2iar\mu - a^2; \quad \mu \equiv \cos \theta, \quad (9)$$

where θ is the colatitude of the considered point, $\mu r = z$, and

$$\Phi_c = \Phi(\sqrt{r^2 - 2iaz - a^2}), \quad (10)$$

$$\rho_c = \rho(\sqrt{r^2 - 2iaz - a^2}). \quad (11)$$

Note that, when starting from a spherically symmetric seed system, the real and imaginary parts of the shifted density ρ_c can be obtained (1) from evaluation of the Laplace operator applied to the real and imaginary parts of the potential Φ_c , (2) by expansion of the complexified density in equation (11), or finally (3) by considering that for spherically symmetric systems $\rho = \rho(\Phi)$, and so the real and imaginary parts of $\rho_c(\Phi_c)$ can be expressed (at least in principle) as functions of the real and imaginary parts of the shifted potential.

2.2 The shifted Plummer sphere

As a first example, in this section we apply the complex shift to the Plummer (1911) sphere. We start from the relative potential $\Psi = -\Phi$, where

$$\Psi = \frac{GM}{b} \frac{1}{\zeta}, \quad \zeta \equiv \sqrt{1+r^2}, \quad (12)$$

and r is normalized to the model scale-length b , so that the associated density distribution is

$$\rho = \frac{3M}{4\pi b^3} \frac{1}{\zeta^5} \quad (13)$$

² If one adopted the standard inner product over the complex field, one would obtain $|\mathbf{x} - i\mathbf{a}|^2 = r^2 + |\mathbf{a}|^2$.

(e.g. see BT). For ease of notation, from hereon we will use the normalized density (to M/b^3) and potential (to GM/b), and so

$$\rho_c = \frac{3}{4\pi} [\Re(\Psi_c) + i\Im(\Psi_c)]^5. \quad (14)$$

From substitution (8) (where the shift a is also expressed in units of b), the shifted potential $\Psi_c = 1/\zeta_c$ depends on the square root of $\zeta_c^2 = 1 - a^2 + r^2 - 2i az \equiv de^{i\varphi}$, with

$$d \equiv |\zeta_c|^2 = \sqrt{(1 - a^2 + r^2)^2 + 4a^2 z^2}, \quad (15)$$

and

$$\cos \varphi = \frac{1 - a^2 + r^2}{d}, \quad \sin \varphi = -\frac{2az}{d}. \quad (16)$$

Note that $\cos \varphi > 0$ everywhere for $a < 1$, and the following discussion is restricted to this case. The square root

$$\zeta_c = \sqrt{d} e^{\pi k i + \varphi i / 2} \quad (k = 0, 1) \quad (17)$$

is made a single-valued function of (r, z) by cutting the complex plane along the negative real axis (which is never touched by ζ_c^2) and assuming $k = 0$, so that the model equatorial plane is mapped into the line $\varphi = 0$. With this choice, the principal determination of Ψ_c reduces to Ψ when $a = 0$, and simple algebra shows that

$$\cos \frac{\varphi}{2} = \frac{\sqrt{1 + \cos \varphi}}{\sqrt{2}}, \quad \sin \frac{\varphi}{2} = -\frac{\sqrt{2}az}{d\sqrt{1 + \cos \varphi}}. \quad (18)$$

The real and imaginary parts of Ψ_c are then given by

$$\Re(\Psi_c) = \Re\left(\frac{\bar{\zeta}_c}{|\zeta_c|^2}\right) = \frac{\sqrt{d+1-a^2+r^2}}{\sqrt{2}d}, \quad (19)$$

$$\Im(\Psi_c) = \Im\left(\frac{\bar{\zeta}_c}{|\zeta_c|^2}\right) = \frac{az}{d^2 \Re(\Psi_c)}, \quad (20)$$

where $\bar{\zeta}_c$ is the complex conjugate of ζ_c , and from equation (14) we obtain the expressions for the (normalized) axisymmetric densities:

$$\Re(\rho_c) = \frac{3\Re(\Psi_c)}{4\pi} \left[\Re(\Psi_c)^4 - \frac{10a^2 z^2}{d^4} + \frac{5a^4 z^4}{d^8 \Re(\Psi_c)^4} \right], \quad (21)$$

$$\Im(\rho_c) = \frac{3\Im(\Psi_c)}{4\pi} \left[5\Re(\Psi_c)^4 - \frac{10a^2 z^2}{d^4} + \frac{a^4 z^4}{d^8 \Re(\Psi_c)^4} \right]. \quad (22)$$

We verified that the two new density–potential pairs (19)–(21) and (20)–(22) satisfy the Poisson equation (1).

Note that, in contrast to $\Re(\Psi_c)$, the potential $\Im(\Psi_c)$ changes sign crossing the equatorial plane of the system. In accordance with this change of sign, $\Im(\rho_c)$ is negative for $z < 0$, as can be seen from equation (22), and thus cannot be used to describe a gravitational system. We do not discuss this pair any further but instead focus on the real components of the shifted density–potential pair. Near the centre $d \sim 1 - a^2 + r^2 [1 + 2a^2 \mu^2 / (1 - a^2)] + O(\mu^2 r^4)$, and the leading terms of the asymptotic expansion of equations (19) and (21) are

$$\Re(\Psi_c) \sim \frac{1}{\sqrt{1-a^2}} - \frac{r^2(1-a^2+3a^2\mu^2)}{2(1-a^2)^{5/2}} + O(r^4), \quad (23)$$

$$\Re(\rho_c) \sim \frac{3}{4\pi(1-a^2)^{5/2}} - \frac{15r^2(1-a^2+7a^2\mu^2)}{8\pi(1-a^2)^{9/2}} + O(r^4); \quad (24)$$

in particular, the isodensities are oblate ellipsoids with a minor-to-major squared axis ratio of $(1 - a^2)/(1 + 6a^2)$. For $r \rightarrow \infty$, $d \sim r^2 + 1 - a^2 + 2a^2 \mu^2 + O(\mu^2 r^{-2})$, and

$$\Re(\Psi_c) \sim \frac{1}{r} - \frac{1 - a^2 + 3a^2 \mu^2}{2r^3} + O(r^{-5}), \quad (25)$$

$$\Re(\rho_c) \sim \frac{15}{4\pi r^5} \left(\frac{1}{5} - \frac{1 - a^2 + 7a^2 \mu^2}{2r^2} \right) + O(r^{-9}), \quad (26)$$

so that $\Re(\rho_c)$ coincides with the unshifted seed density (13) and is spherically symmetric and positive.³ Thus, near the centre and in the far field $\Re(\rho_c) > 0$ for $0 \leq a < 1$. In addition, on the model equatorial plane $z = 0$ (where $d = 1 - a^2 + R^2$, with R the cylindrical radius), $\Re(\Psi_c)$ coincides with the potential of a Plummer sphere of scale-length $\sqrt{1 - a^2}$, and from equation (21) it follows that $\Re(\rho_c) = 3\Re(\Psi_c)^5 / (4\pi) > 0$ for $0 \leq a < 1$.

However, for $z \neq 0$ a negative term is present in equation (21), and the positivity of $\Re(\rho_c)$ is not guaranteed for a generic value of the shift parameter in the range $0 \leq a < 1$. In fact, a numerical exploration reveals that $\Re(\rho_c)$ becomes negative on the symmetry axis $R = 0$ at $z \simeq 0.81$ for $a = a_m \simeq 0.588$; the negative-density region then expands around this critical point for increasing $a > a_m$. The isodensity contours of $\Re(\rho_c)$ in a meridional plane are shown in the top panels of Fig. 1, for shift-parameter values $a = 1/2$ (left) and $a = 23/40 = 0.575$ (right). The most salient property is the resulting toroidal shape of the model with the large shift, which is reminiscent of other similar structures known in the literature, for example the Lynden-Bell (1962) flattened Plummer sphere, the densities associated with the Binney (1981) logarithmic potential and the Evans (1994) scale-free potentials, the Toomre (1982) and Ciotti & Bertin (2005) tori (see also Ciotti, Bertin & Londrillo 2004), and the exact Modified Newtonian Dynamics density–potential pairs discussed in Ciotti, Nipoti & Londrillo (2006). Unfortunately, we were not able to find an explanation (if indeed any exists) for this similarity (see, however, Section 4).

2.3 The shifted isochrone sphere

Following the treatment of the Plummer sphere, we now consider the slightly more complicated case of the shifted isochrone sphere. Its relative potential and density are given by

$$\Psi = \frac{GM}{b} \frac{1}{1 + \zeta}, \quad (27)$$

$$\rho = \frac{M}{4\pi b^3} \frac{3(1 + \zeta) + 2r^2}{(1 + \zeta)^3 \zeta^3}, \quad (28)$$

where ζ is defined in equation (12), and again all lengths are normalized to the scale b (Hénon 1959; see also BT). In analogy with equation (14), the normalized shifted density can be written as a function of the normalized shifted potential:

$$\rho_c = \frac{\Psi_c^2}{4\pi \zeta_c^3} (3 + 2r_c^2 \Psi_c). \quad (29)$$

The real and imaginary parts of $\Psi_c = (1 + \bar{\zeta}_c) / [1 + |\zeta_c|^2 + 2\Re(\zeta_c)]$ are easily obtained from equations (15)–(18) as

$$\Re(\Psi_c) = \frac{1 + \sqrt{d+1-a^2+r^2}/\sqrt{2}}{1+d+\sqrt{2(d+1-a^2+r^2)}}, \quad (30)$$

$$\Im(\Psi_c) = \frac{\sqrt{2}az}{\sqrt{d(d+1-a^2+r^2)[1+d+\sqrt{2(d+1-a^2+r^2)}]}}, \quad (31)$$

³ This is a general property of shifted spherical systems, as can be seen by expanding equation (11) for $r \rightarrow \infty$.

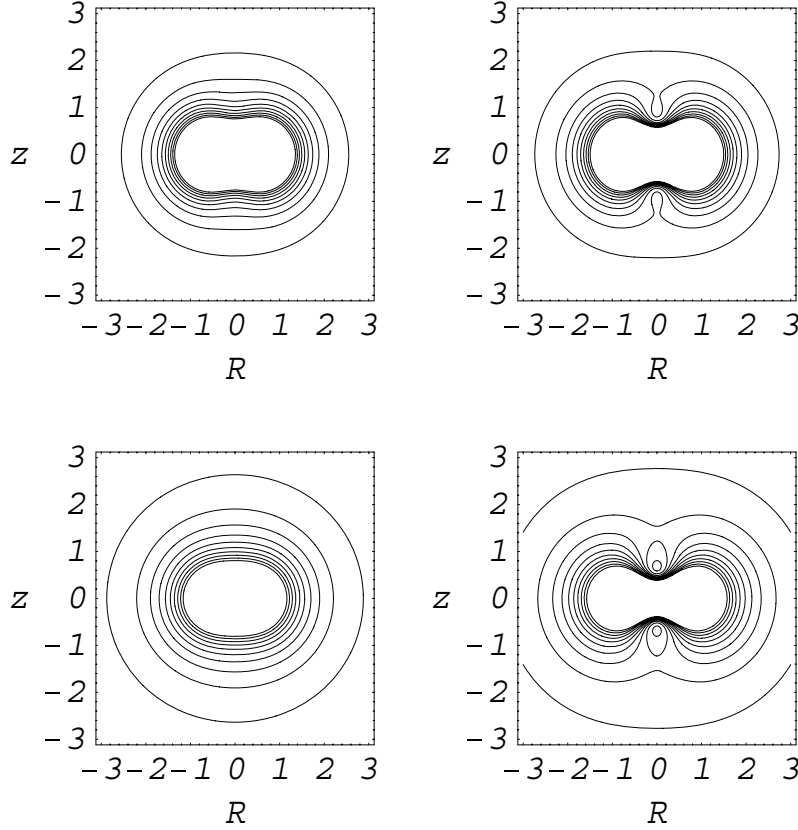


Figure 1. Isodensity contours in the (R, z) plane of $\mathfrak{R}(\rho_c)$ of the shifted Plummer sphere for $a = 1/2$ (top left) and $a = 23/40$ (top right), and of the shifted isochrone sphere for $a = 1/2$ (bottom left) and $a = 4/5$ (bottom right). The coordinates are normalized to the scale-length b of the corresponding seed spherical model.

while $\mathfrak{R}(\rho_c)$ and $\mathfrak{Z}(\rho_c)$ can be obtained by expansion of equation (29); however, their expression is quite cumbersome, and so is not detailed here. Again, $\mathfrak{Z}(\Psi_c)$ changes sign when crossing the model equatorial plane, revealing that $\mathfrak{Z}(\rho_c)$ also changes sign in order to produce the resulting vertical force field near $z = 0$, and the vanishing of the integral in equation (7). In accordance with equation (11) on the model equatorial plane, $\mathfrak{R}(\rho_c)$ is functionally identical to an isochrone sphere of scale-length $\sqrt{1 - a^2}$ (and so is positive for $0 \leq a < 1$), while for $r \rightarrow \infty$

$$\mathfrak{R}(\Psi_c) \sim \frac{r-1}{r^2} + \frac{1+a^2-3a^2\mu^2}{2r^3} + O(r^{-4}), \quad (32)$$

$$\mathfrak{R}(\rho_c) \sim \frac{1}{4\pi r^5} \left(2r - 3 + 4a^2 \frac{1-6\mu^2}{r} \right) + O(r^{-7}). \quad (33)$$

As for the shifted Plummer sphere, it is also the case here that $\mathfrak{R}(\rho_c)$ coincides in the far field with the seed density, and so is positive for $0 \leq a < 1$, while it becomes negative on the z axis at $z \simeq 0.648$ for $a > a_m \simeq 0.804$. In the bottom panels of Fig. 1 we show the isodensity contours in the meridional plane of $\mathfrak{R}(\rho_c)$ for the representative values of the shift parameter $a = 1/2$ (left) and $a = 4/5$ (right): again, the toroidal shape and the critical regions on the symmetry axis are apparent in the case of the larger shift.

3 ROTATIONAL FIELDS

A natural question to ask concerns the nature of the kinematical fields that can be supported by the found density–potential pairs

$\mathfrak{R}(\Psi_c) - \mathfrak{R}(\rho_c)$ if they represent self-gravitating stellar systems. In general, the associated two-integral Jeans equations (e.g. BT, Ciotti 2000) can be solved numerically after having fixed the relative amount of ordered streaming motions and velocity dispersion in the azimuthal direction (e.g. see Satoh 1980; Ciotti & Pellegrini 1996). For simplicity, we restrict our attention here to the isotropic case, in which the resulting Jeans equations are formally identical to the equations describing axisymmetric, self-gravitating gaseous systems in permanent rotation:

$$\begin{cases} \frac{1}{\rho} \frac{\partial P}{\partial z} = -\frac{\partial \Phi}{\partial z}, \\ \frac{1}{\rho} \frac{\partial P}{\partial R} = -\frac{\partial \Phi}{\partial R} + \Omega^2 R \end{cases} \quad (34)$$

(in order to simplify the notation, in the following we intend $\rho = \mathfrak{R}[\rho_c]$ and $\Phi = \mathfrak{R}[\Phi_c]$). The quantities ρ , P and Ω denote the fluid density, pressure and angular velocity, respectively; the rotational (i.e. streaming) velocity is $v_\phi = \Omega R$, while $v_R = v_z = 0$.

In problems in which ρ and ϕ are assigned, the standard approach for the solution of equations (34) is to integrate for the pressure with boundary condition $P(R, \infty) = 0$,

$$P = \int_z^\infty \rho \frac{\partial \Phi}{\partial z'} dz', \quad (35)$$

and then to obtain the rotational velocity field from the radial equation

$$v_\phi^2 = R \frac{\partial P}{\partial R} + R \frac{\partial \Phi}{\partial R}. \quad (36)$$

However, v_φ^2 can also be obtained without previous knowledge of P , because by combining equations (35)–(36) and integrating by parts with the assumption that $P = \rho \partial \Phi / \partial R = 0$ for $z = \infty$ it follows that

$$\frac{\rho v_\varphi^2}{R} = \int_z^\infty \left(\frac{\partial \rho}{\partial R} \frac{\partial \Phi}{\partial z'} - \frac{\partial \rho}{\partial z'} \frac{\partial \Phi}{\partial R} \right) dz'. \quad (37)$$

We note that this ‘commutator-like’ relationship is not new (for example see Rosseland 1926; Waxman 1978, and, in the context of stellar dynamics, Hunter 1977).

Before solving equations (34) for the real parts of the density–potential pairs of Section 2, it is useful to recall some basic properties of rotating fluid configurations. For example, in several astrophysical applications (for example the set-up of initial conditions for hydrodynamical simulations), equations (34) are solved for the density under the assumption of a barotropic pressure distribution (and neglecting the gas self-gravity), and for assigned $\rho(R, \infty) = 0$ or $\rho(R, 0) = \rho_0(R)$.

As is well known, this approach can lead only to hydrostatic ($\Omega = 0$) or cylindrical rotation ($\Omega = \Omega[R]$) kinematical fields. In fact, according to the Poincaré–Wavre theorem (e.g. see Lebovitz 1967; Tassoul 1978), cylindrical rotation is *equivalent* to barotropicity, or to the fact that the acceleration field on the right-hand side of equations (34) derives from the effective potential

$$\Phi_{\text{eff}} \equiv \Phi - \int_{R_0}^R \Omega^2(R') R' dR' \quad (38)$$

(where R_0 is an arbitrary but fixed radius), so that the gas density and pressure are stratified on Φ_{eff} .

However, barotropic equilibria are just a very special class of solutions of equations (34). For example, in the context of galactic dynamics isotropic axisymmetric galaxy models often show streaming velocities dependent on z (for simple examples see, for example, Lanzoni & Ciotti 2003; Ciotti & Bertin 2005). Such baroclinic configurations (i.e. fluid systems in which P cannot be expressed as a function of ρ only, and Φ_{eff} does not exist at all) have been studied in the past for problems ranging from geophysics, to the theory of sunspots and stellar rotation (e.g. see Rosseland 1926), to the problem of modelling the decrease of rotational velocity of the extraplanar gas in disc galaxies for increasing z (e.g. see Barnabé et al. 2005, 2006). Note that a major problem posed by the construction of baroclinic solutions is the fact that the existence of physically acceptable solutions (i.e. configurations for which $v_\varphi^2 \geq 0$ everywhere) is not guaranteed for arbitrary choices of ρ and Φ . However, the positivity of the integrand in equation (37) for $z \geq 0$ is a sufficient condition to have $v_\varphi^2 \geq 0$ everywhere. In fact, in Barnabé et al. (2006) several theorems on baroclinic configurations have been proved starting from equation (37): in particular, *toroidal* gas distributions are strongly favoured in order to have $v_\varphi^2 \geq 0$ in the presence of a dominating disc gravitational field. We note that also in the present cases the density is of toroidal shape, although the distribution is self-gravitating.

In the case of the $\mathfrak{R}(\rho_c)$ – $\mathfrak{R}(\Psi_c)$ pair of the shifted Plummer sphere, the (normalized) commutator in equation (37) is a very simple function:

$$\frac{\partial \rho}{\partial R} \frac{\partial \Phi}{\partial z'} - \frac{\partial \rho}{\partial z'} \frac{\partial \Phi}{\partial R} = \frac{15a^2}{\pi} \frac{Rz(1 - a^2 + r^2)}{d^7}, \quad (39)$$

which is positive for $z > 0$ and $0 \leq a < 1$, so that $v_\varphi^2 \geq 0$ as far as $\mathfrak{R}(\rho_c)$ is positive everywhere, i.e. for $0 \leq a < a_m$. Remarkably,

the integration of equation (37) can be easily carried out in terms of elementary functions:

$$\rho v_\varphi^2 = \frac{3a^2 R^2}{2\pi d^5} \left\{ 1 - \frac{d^4(1 + a^2 + r^2)}{12a^4(1 + R^2)^3} \times \left[\frac{6a^4(1 + R^2)^2}{d^4} - \frac{2a^2(1 + R^2)}{d^2} + \frac{1 + a^2 + r^2 - d}{1 + a^2 + r^2} \right] \right\}, \quad (40)$$

while the direct integration of the pressure equation is much more complicated [again showing the relevance of equation (37)], and we report its normalized expression on the equatorial plane only:

$$P(R, 0) = \frac{2(1 + R^2) - a^2}{16\pi(1 + R^2)^2(1 - a^2 + R^2)^2}. \quad (41)$$

Note that $v_\varphi^2 \propto a^2$, and so it vanishes (as a consequence of the imposed isotropy) when reducing to the spherically symmetric seed density: in fact, it can be proved that the argument in parentheses in equation (40) is regular for $a \rightarrow 0$. The baroclinic nature of the equilibrium is apparent, and confirmed by Fig. 2, where we show the contours of constant v_φ^2 for $a = 1/2$ and $a = 23/40$ (top panels, corresponding to the models plotted in the top panels of Fig. 1). The normalized streaming velocity field in the equatorial plane is

$$v_\varphi^2(R, 0) = \frac{a^2 R^2}{3} \frac{6(1 + R^2)^2 - 4a^2(1 + R^2) + a^4}{(1 + R^2)^3(1 - a^2 + R^2)^{3/2}}, \quad (42)$$

and has a maximum at $R \lesssim 1$, while the normalized circular velocity is the same as that of a Plummer sphere of scale-length $\sqrt{1 - a^2}$ [cf. the comment after equation (26)]; that is,

$$v_c^2(R) = R \frac{\partial \Phi(R, 0)}{\partial R} = \frac{R^2}{(1 - a^2 + R^2)^{3/2}}. \quad (43)$$

It is easy to verify that equations (41) and (42) satisfy equation (36).

Unfortunately, we were unable to obtain the explicit formulae for the analogous quantities for the shifted isochrone model. However, it is easy to prove that also in this case the commutator is proportional to a^2 , and so vanishes (as expected) for $a \rightarrow 0$. A comparison of the rotational fields of the two models in the far field can be obtained by using the asymptotic expressions (25)–(26) and (32)–(33). In particular, for the real part of the shifted Plummer sphere we found

$$v_\varphi^2 = \frac{2a^2 R^2}{r^5} - \frac{5a^2 R^2(3 - a^2 - 9a^2 \mu^2)}{3r^7} + O(R^2 r^{-9}), \quad (44)$$

while for $\mathfrak{R}(\rho_c)$ of the isochrone shifted sphere

$$v_\varphi^2 = \frac{a^2 R^2}{r^5} \left(\frac{4}{3} - \frac{13}{5r} \right) + \frac{a^2 R^2(3159 + 5780a^2 + 21780a^2 \mu^2)}{r^7} + O(R^2 r^{-8}). \quad (45)$$

Note that, even though the radial asymptotic behaviour of the density is different in the two cases considered, the obtained velocity fields have the same radial dependence in the far field, and in both cases v_φ^2 decreases for increasing z and fixed R . In addition, $v_\varphi^2 \rightarrow 0$ for $r \rightarrow \infty$, as expected from the spherical symmetry of the density and potential for $r \rightarrow \infty$: in fact, it is apparent from equation (37) that spherical (additive) components of ρ and Φ [for example the leading terms in the asymptotic expansions (25)–(26) and (32)–(33)] mutually cancel in the commutator evaluation.

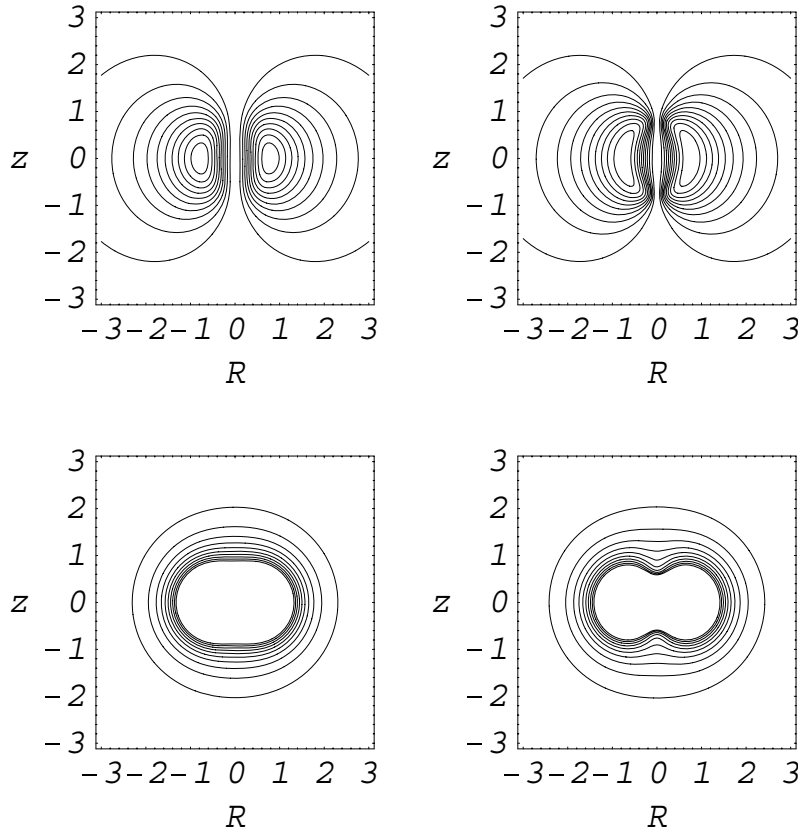


Figure 2. Isorotational contours in the meridional plane of $\Re(\rho_c)$ of the shifted Plummer sphere for $a = 1/2$ (top left), and $a = 23/40$ (top right). The bottom panels show contours of constant pressure for the same models.

4 CONCLUSIONS

In this paper we have shown that the complex-shift method, introduced in electrodynamics by Newman, and thoroughly studied by, among others, Carter, Lynden-Bell and Kaiser, can be extended to classical gravitation to produce new and explicit density–potential pairs with finite deviation from spherical symmetry. In particular, we showed that, after separation of the real and imaginary parts of the complexified density–potential pair, the imaginary part of the density corresponds to a system of null total mass, while the real component can be positive everywhere (depending on the original seed density distribution and the size of the complex shift).

As a simple application of the proposed method we examined the properties of the axisymmetric systems resulting from the shift of the Plummer and isochrone spheres. The analysis revealed that only for shift values in some restricted range are the obtained densities nowhere negative. This property is linked to the presence of a flat ‘core’ in the adopted (spherically symmetric) seed distributions. It is thus expected that other distributions, such as the King (1972) model, and the Dehnen (1993) and Tremaine et al. (1994) $\gamma = 0$ model, will also lead to physically acceptable shifted systems. For the two new exact density–potential pairs we also found that the associated rotational fields (obtained under the assumption of global isotropy of the velocity dispersion tensor) correspond to baroclinic gaseous configurations.

An interesting issue raised by the present analysis is the toroidal shape of the obtained densities. While we were not able to find a general explanation for this phenomenon, some hints can be obtained by considering the behaviour of the complex shift for $a \rightarrow 0$.

In fact, in this case it can easily be proved that, while the *odd* terms of the expansion correspond to the imaginary part of the shifted density (and contain the factor z), the *even* terms are real and, to second order in a ,

$$\Re(\rho_c) \sim \rho - \frac{a^2}{2r} \frac{d\rho}{dr} + \frac{a^2 z^2}{2r^3} \left(\frac{d\rho}{dr} - r \frac{d^2\rho}{dr^2} \right) + O(a^4). \quad (46)$$

This expansion is functionally similar to that obtained by Ciotti & Bertin (2005) for the case of oblate homeoidal distributions, where the term dependent on z^2 is immediately identified with a spherical harmonic (e.g. Jackson 1999) responsible for the toroidal shape. It is clear that, when restricting attention to the small-shift approximation (46), all the computations presented in this paper can be carried out explicitly for simple seed distributions. Moreover, from equation (46) it is easy to prove that negative values of density appear on the z -axis – no matter how small the shift parameter is – when the seed density has a central cusp $\rho \propto r^{-\gamma}$ with $\gamma > 0$ (see, for example, the explicit example in the Appendix).

For simplicity, we restricted our analysis in this paper to spherically symmetric seed systems, which produce axisymmetric systems. A natural extension of the present investigation would be the study of the density–potential pairs originating from the complex shift of seed disc distributions, such as the Miyamoto & Nagai (1975) and Satoh (1980) discs, or even from the generalization of homeoidal quadrature formulae (see, for example, Kellogg 1953; Chandrasekhar 1969, BT). Note that in such cases, in contrast to the spherical cases discussed here, the shift *direction* is important: for example, the complex shift of a disc along the z -axis will lead

to a different system from that of an equatorial shift, which would produce a triaxial object.

A second interesting line of study could be the use of the complex shift to produce more elaborate density–potential pairs by adding models with a weight function $w(\mathbf{a})$; that is, by considering the behaviour of the linear operator

$$\varrho_c = \int \rho(\mathbf{x} - i\mathbf{a})w(\mathbf{a})d^3\mathbf{a}, \quad (47)$$

where the integration is extended over some suitably chosen region. We have not pursued this line of investigation, but it is obvious that shifted density components with negative regions can be accepted in this context, as long as it is possible to construct a weight function leading to a final distribution nowhere negative (as happens, for example, in spherical harmonics expansions).

ACKNOWLEDGMENTS

The paper is dedicated to the memory of my best friend and colleague Dr Giacomo Giampieri, who tragically passed away on 2006 September 3, while working on this project. LC thanks Donald Lynden-Bell for enlightening discussions, and Tim de Zeeuw, Scott Tremaine and an anonymous referee for interesting comments. This work was supported by the Italian MIUR grant CoFin2004 ‘Collective phenomena in the dynamics of galaxies’.

REFERENCES

- Appell P., 1887, *Ann. Math. Lpz.*, 30, 155
 Barnabé M., Ciotti L., Fraternali F., Sancisi R., 2005, in Braun R., ed., *ASP Conf. Ser. Vol. 331, Extra-planar Gas. Astron. Soc. Pac., San Francisco*, p. 231
 Barnabé M., Fraternali F., Ciotti L., Sancisi R., 2006, *A&A*, 446, 61
 Binney J., 1981, *MNRAS*, 196, 455
 Binney J., Tremaine S., 1987, *Galactic Dynamics*. Princeton Univ. Press Princeton (BT)
 Carter B., 1968, *Commun. Math. Phys.*, 10, 280
 Chandrasekhar S., 1969, *Ellipsoidal Figures of Equilibrium*. Yale Univ. Press, New Haven
 Chandrasekhar S., 1976, *Proc. R. Soc. Lond. A*, 349, 571
 Ciotti L., 2000, *Lectures Notes on Stellar Dynamics*. Scuola Normale Superiore, Pisa
 Ciotti L., Bertin G., 2005, *A&A*, 437, 419
 Ciotti L., Pellegrini S., 1996, *MNRAS*, 279, 240
 Ciotti L., Bertin G., Londrillo P., 2004, in Bertin G., Farina D., Pozzoli R., eds, *Plasmas in the Laboratory and in the Universe: New insights and New Challenges*, AIP Conf. Proc., 703, 322
 Ciotti L., Nipoti C., Londrillo P., 2006, *ApJ*, 640, 741
 D’Afonseca L. A., Letelier P. S., Oliveira S. R., 2005, *Class. Quant. Grav.*, 22, 3803
 Dehnen W., 1993, *MNRAS*, 265, 250
 de Zeeuw P. T., Pfenniger D., 1988, *MNRAS*, 235, 949
 de Zeeuw P. T., Carollo C. M., 1996, *MNRAS*, 281, 1333
 Evans N. W., 1994, *MNRAS*, 267, 333
 Evans N. W., de Zeeuw P. T., 1992, *MNRAS*, 257, 152
 Ferrers N. M., 1877, *Quart. J. Pure Appl. Math.*, 14, 1
 Gleiser R., Pullin J., 1989, *Class. Quant. Grav.*, 6, 977
 Hénon M., 1959, *Ann. d’Astrophys.*, 22, 126
 Hunter C., 1977, *AJ*, 82, 271
 Jackson J. D., 1999, *Classical Electrodynamics*, 3rd edn. Wiley, New York
 Jeffreys H., 1970, *The Earth*, 5th edn. Cambridge Univ. Press, Cambridge
 Kaiser G., 2004, *J. Phys. A: Math. Gen.*, 37, 8735
 Kellogg O. D., 1953, *Foundations of Potential Theory*. Dover, New York
 King I., 1972, *ApJ*, 174, L123
 Kutuzov S. A., 1998, *Astron. Lett.*, 24, 645
 Kutuzov S. A., Osipkov L. P., 1980, *AZh*, 57, 28
 Lanzoni B., Ciotti L., 2003, *A&A*, 404, 819
 Lebovitz N. R., 1967, *ARA&A*, 5, 465
 Lee J., Suto Y., 2003, *ApJ*, 585, 151
 Lee J., Suto Y., 2004, *ApJ*, 601, 599
 Letelier P. S., Oliveira S. R., 1987, *J. Math. Phys.*, 28, 165
 Letelier P. S., Oliveira S. R., 1998, *Class. Quant. Grav.*, 15, 421
 Lynden-Bell D., 1962, *MNRAS*, 123, 447
 Lynden-Bell D., 2000, *MNRAS*, 312, 301
 Lynden-Bell D., 2002, preprint (astro-ph/0207064)
 Lynden-Bell D., 2004a, *Phys. Rev. D*, 70, 104021
 Lynden-Bell D., 2004b, *Phys. Rev. D*, 70, 105017
 Miyamoto M., Nagai R., 1975, *PASJ*, 27, 533
 Muccione V., Ciotti L., 2003, in Contopoulos G., Voglis N., eds, *Galaxies and Chaos. Lecture Notes on Physics*, 626, 387
 Muccione V., Ciotti L., 2004, *A&A*, 421, 583
 Newman E. T., 1973, *J. Math. Phys.*, 14, 102
 Newman E. T., Janis A. I., 1965, *J. Math. Phys.*, 6, 915
 Newman E. T., Couch E. C., Chinnappared K., Exton A., Prakash A., Torrence R., 1965, *J. Math. Phys.*, 6, 918
 Page D. N., 1976, *Phys. Rev. D*, 14, 1509
 Pfenniger D., 1984, *A&A*, 134, 373
 Plummer H. C., 1911, *MNRAS*, 71, 460
 Rosselland S., 1926, *ApJ*, 63, 342
 Satoh C., 1980, *PASJ*, 32, 41
 Tassoul J. L., 1978, *Theory of Rotating Stars*. Princeton Univ. Press, Princeton
 Teukolsky S., 1973, *ApJ*, 185, 635
 Toomre A., 1982, *ApJ*, 259, 535
 Tremaine S., Richstone D. O., Byun Y. I., Dressler A., Faber S. H., Grillmair C., Kormendy J., Laver T. R., 1994, *AJ*, 107, 634
 Waxman A. M., 1978, *ApJ*, 222, 61
 Whittaker E. T., Watson G. N., 1950, *A Course of Modern Analysis*. Cambridge Univ. Press, Cambridge

APPENDIX A: THE SHIFTED SINGULAR ISOTHERMAL SPHERE

In this case the normalized density and potential are $\rho = 1/r^2$ and $\Phi = 4\pi \ln r$, so that from equations (10)–(11)

$$\rho_c = \frac{r^2 - a^2}{(r^2 - a^2)^2 + 4a^2z^2} + i \frac{2az}{(r^2 - a^2)^2 + 4a^2z^2}, \quad (A1)$$

$$\Phi_c = 2\pi \ln(r^2 - a^2 - 2iaz). \quad (A2)$$

Note how, in contrast to the Plummer and isochrone spheres, $\Re(\rho_c)$ here describes a density distribution negative inside the open ball $r < a$ and positive outside. The density on the surface $r = a$ is zero except on the singular ring $r = a$ in the equatorial plane. In the far field the density decreases as $1/r^2$, while near the centre it flattens to $-1/a^2$, while the potential generated by $\Re(\rho_c)$ is given by

$$\Re(\Phi_c) = \pi \ln[(r^2 - a^2)^2 + 4a^2z^2]. \quad (A3)$$

As for the other two cases discussed in this paper, the density $\Im(\rho_c)$ vanishes on the equatorial plane, and is positive above it and negative below.

This paper has been typeset from a $\text{\TeX}/\text{\LaTeX}$ file prepared by the author.