

Consistency criteria for generalized Cuddeford systems

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ABSTRACT

General criteria to check the positivity of the distribution function (phase-space consistency) of stellar systems of assigned density and anisotropy profile are useful starting points in Jeans-based modelling. Here, we substantially extend previous results, and present the inversion formula and the analytical necessary and sufficient conditions for phase-space consistency of the family of multicomponent Cuddeford spherical systems: the distribution function of each density component of these systems is defined as the sum of an arbitrary number of Cuddeford distribution functions with arbitrary values of the anisotropy radius, but identical angular momentum exponent. The radial trend of anisotropy that can be realized by these models is therefore very general. As a surprising byproduct of our study, we found that the ‘central cusp–anisotropy theorem’ (a necessary condition for consistency relating the values of the central density slope and of the anisotropy parameter) holds not only at the centre but also at all radii in consistent multicomponent generalized Cuddeford systems. This last result suggests that the so-called mass–anisotropy degeneracy could be less severe than what is sometimes feared.

Key words: stellar dynamics – celestial mechanics – galaxies: kinematics and dynamics.

1 INTRODUCTION

In the study of stellar systems based on the ‘ ρ -to- f ’ approach [where ρ is the material density and f is the associated phase-space distribution function (DF); e.g. see Bertin 2000; Binney & Tremaine 2008], the density distribution is given, and specific assumptions on the internal dynamics of the model are made. In some special cases, inversion formulae exist so that the DF can be obtained, usually in integral form or as series expansion (see e.g. Fricke 1952; Lynden Bell 1962; Osipkov 1979; Merritt 1985; Dejonghe 1986, 1987; Cuddeford 1991; Hunter & Qian 1993; Ciotti & Bertin 2005). Once the DF of the system is derived, a non-negativity check is (or should be) performed, and in case of negative values the model must be discarded as unphysical. Indeed, a minimal but essential requirement to be met by the DF (of each component) of a stellar dynamical model is positivity over the accessible phase-space. This requirement, the so-called phase-space consistency, is much weaker than the model stability, but it is stronger than the fact that the Jeans equations have a physically acceptable solution. However, the difficulties inherent in the operation of recovering analytically the DF prevent in general a simple consistency analysis, and numerical inspection of the inversion integral is required. As a consequence, the reasons underlying consistency or inconsistency of a proposed model are

somewhat obscured by the numerical nature of the solution. Fortunately, criteria for phase-space consistency that can be applied without an explicit recovery of the DF are known and widely used. For example, analytical necessary and sufficient conditions for consistency of multicomponent systems with Osipkov–Merritt (OM) anisotropy (Osipkov 1979; Merritt 1985) were derived in Ciotti & Pellegrini (1992, hereafter CP92; see also Tremaine et al. 1994) and applied in several investigations (e.g. Ciotti 1996, hereafter C96; Ciotti 1999, hereafter C99; Ciotti & Lanzoni 1997; Ciotti & Morganti 2009, hereafter CM09; Ciotti, Morganti & de Zeeuw 2009). Such conditions revealed not only simple and useful to investigate the phase-space consistency of OM models but also helpful to elucidate the different roles of total potential, orbital anisotropy and stellar and dark matter density profiles in making a model unphysical.

More recently, the ‘central cusp–anisotropy theorem’ (An & Evans 2006, hereafter AE06), a necessary condition (NC) for consistency relating the values of the central density slope and of the anisotropy parameter β (see equation 4) has been proved.¹ This condition was derived for constant anisotropy systems, and then generalized asymptotically to the central regions of spherical systems with arbitrary anisotropy distribution. A remarkable property of the density slope–anisotropy inequality is that it actually holds rigorously at *every* radius in constant anisotropy systems, and not

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¹The same inequality was also reported in equation (28) in de Bruijne, van der Marel & de Zeeuw (1996).

only at their centre (AE06). Surprisingly, in CM09 we showed that the CP92 NC for model consistency can be formally rewritten as the AE06 inequality, that consequently holds at each radius not only in constant anisotropy systems but also in multicomponent OM systems! This curious result prompted us to investigate the phase-space consistency of Cuddeford (1991) anisotropic systems, as they generalize both constant and OM anisotropy and an explicit inversion formula exists, so that necessary and sufficient conditions for consistency can hopefully be found, extending those of CP92. In addition, Cuddeford anisotropy allows to explore systems in which the central regions may be tangentially anisotropic, at variance with the OM cases.

Actually, we found it possible to extend our study to the very general case of multicomponent, generalized Cuddeford systems, i.e. spherical systems in which the DF of *each* distinct density component is assumed to be the sum of an arbitrary number of Cuddeford DFs with arbitrarily different anisotropy radii but identical angular momentum exponent (see equation 22). In this paper, we show how the family of necessary and sufficient conditions for model consistency can be derived for generalized Cuddeford anisotropic systems. We also found that the first of the NCs coincides again with the density slope–anisotropy theorem, thus demonstrating that such inequality must be satisfied at all radii also in the whole family of consistent, multicomponent generalized Cuddeford systems.

The paper is organized as follows. In Section 2, we recall the fundamental properties of OM and constant anisotropy systems, and the associated consistency criteria. In Section 3, we derive the family of consistency criteria for the larger class of multicomponent galaxy models with generalized Cuddeford anisotropy. Then, in Section 4 some illustrative applications of the new phase-space consistency criteria are presented, and in Section 5 the main results are summarized, with a brief discussion of the relevance of the new findings for the mass–anisotropy degeneracy problem. In the Appendix, we prove that the first of the NCs for phase-space consistency of multicomponent generalized Cuddeford systems can be rewritten as the density slope–anisotropy inequality, that must hold at all radii.

2 CONSISTENCY CRITERIA FOR MULTICOMPONENT OSIPKOV–MERRITT SYSTEMS

In this section, we summarize the main features of the OM inversion procedure, focusing on the arguments upon which the derivation of the CP92 necessary and sufficient conditions for phase-space consistency is based: in fact, similar arguments will be applied to generalized Cuddeford systems in Section 3.

To fix the notation, we say that a multicomponent stellar system described by a sum of different density components ρ_i is called consistent if each DF f_i is non-negative over the whole accessible phase-space. However, as all the conditions presented in this paper hold for each ρ_i , for simplicity from now on the index i is not indicated, except when required for clarity.

The OM prescription assumes that the DF supporting each density component depends on the energy and on the angular momentum modulus of stellar orbits as

$$f = f(Q), \quad Q = \mathcal{E} - \frac{J^2}{2r_a^2}, \quad (1)$$

and $f(Q) = 0$ for $Q \leq 0$. In the formula above, $\mathcal{E} = \Psi_T - v^2/2$ is the binding energy per unit mass, $\Psi_T = -\Phi_T$, where Φ_T is the potential due to the combined effect of all the components

ρ_i , and r_a is the so-called anisotropy radius of each component (e.g. see Binney & Tremaine 2008). Each density component of a multicomponent OM system is characterized by a DF of the family (1), in general with different r_{ai} : therefore, unless all the r_{ai} are identical a multicomponent OM system is *not* an OM system. It is easy to prove that the DF of each component is related to its density profile as

$$\rho = \int f d^3v = 2\sqrt{8\pi}A(r, r_a) \int_0^{\Psi_T} \sqrt{\Psi_T - Q} f(Q) dQ, \quad (2)$$

where

$$A(r, r_a) = \frac{r_a^2}{r_a^2 + r^2}. \quad (3)$$

The radial dependence of the associated anisotropy parameter, a quantity designed to measure the differences between the tangential (σ_t^2) and radial (σ_r^2) velocity dispersions, is

$$\beta(r) \equiv 1 - \frac{\sigma_t^2}{2\sigma_r^2} = \frac{r^2}{r^2 + r_a^2} \quad (4)$$

(Merritt 1985), so that the orbital distribution is isotropic at the centre and increasingly radially anisotropic with radius. Note that consistency implies $\beta \leq 1$. With the introduction of the so-called ‘augmented density’

$$\varrho(r) \equiv \frac{\rho}{A(r, r_a)} = \left(1 + \frac{r^2}{r_a^2}\right) \rho(r), \quad (5)$$

it is possible to recast equation (2) in a form suitable for Abel inversion, and after the differentiation one obtains

$$\frac{d\varrho}{d\Psi_T} = \sqrt{8\pi} \int_0^{\Psi_T} \frac{f(Q) dQ}{\sqrt{\Psi_T - Q}}, \quad (6)$$

where the function ϱ is intended to be expressed in terms of Ψ_T , by the elimination of radius. As first solved by Eddington (1916) for the isotropic case in which $Q = \mathcal{E}$, equation (6) can be inverted as

$$\begin{aligned} f(Q) &= \frac{1}{\sqrt{8\pi^2}} \frac{d}{dQ} \int_0^Q \frac{d\varrho}{d\Psi_T} \frac{d\Psi_T}{\sqrt{Q - \Psi_T}} \\ &= \frac{1}{\sqrt{8\pi^2}} \int_0^Q \frac{d^2\varrho}{d\Psi_T^2} \frac{d\Psi_T}{\sqrt{Q - \Psi_T}} \end{aligned} \quad (7)$$

(Osipkov 1979), where the second identity above holds for untruncated systems with finite total mass. Equation (6) is also of central importance in the derivation of the CP92 NC.

Theorem: an NC for the non-negativity of the DF of each density component ρ in a multicomponent OM system is

$$\frac{d\varrho}{d\Psi} \geq 0, \quad 0 \leq \Psi \leq \Psi(0), \quad (8)$$

where ϱ is the augmented density in equation (5) and Ψ is the relative gravitational potential of the considered density component. The NC can be rewritten in terms of the logarithmic density slope $\gamma(r) \equiv -d \ln \rho / d \ln r$ and of the anisotropy parameter $\beta(r)$ as

$$\gamma(r) \geq 2\beta(r), \quad \forall r. \quad (9)$$

In addition, a *weak sufficient condition* (WSC) for the non-negativity of each DF is

$$\frac{d}{d\Psi} \left(\frac{d\varrho}{d\Psi_T} \right) \geq 0, \quad 0 \leq \Psi \leq \Psi(0). \quad (10)$$

Proof: see CP92, C96 and CM09. Here, we just recall that the NC is obtained by assuming a positive $f(Q)$ in equation (6), while

the WSC by requiring the positivity of the integrand in the second equation (7), i.e. inequality (10) is nothing else than a rewriting of $d^2\varrho/d\Psi_T^2 \geq 0$.

Of particular relevance for the following discussion is inequality (9), an unexpected extension of the ‘central slope–anisotropy theorem’:

Theorem: in all consistent *constant anisotropy* systems (with $\beta \leq 1/2$) necessarily

$$\gamma(r) \geq 2\beta, \quad \forall r. \quad (11)$$

Moreover, the same inequality holds asymptotically at the centre (i.e. for $r \rightarrow 0$) of any consistent spherical system with *generic anisotropy profile*.

Proof: see section 2.1.1 in AE06.

For completeness, we recall that systems with constant anisotropy are generated assuming a DF of the form

$$f = J^{2\alpha} h(\mathcal{E}), \quad (12)$$

where $h(\mathcal{E})$ is a positive function, and $\alpha > -1$ is a real number (see Section 3; see also Binney & Tremaine 2008). In such models, the anisotropy parameter is

$$\beta(r) = -\alpha, \quad (13)$$

so that for $\alpha > 0$ they are characterized by tangential anisotropy, while for $-1 < \alpha < 0$ the orbital anisotropy is radial. The proof of identity (13) and the inversion formula analogous to (7) are not given here, being obtained as special cases of the Cuddeford systems described in the next section.

3 CONSISTENCY CRITERIA FOR MULTICOMPONENT GENERALIZED CUDDEFORD SYSTEMS

We begin this Section by recalling the main features of the inversion for Cuddeford (1991) systems. Then, in Section 3.2 the family of multicomponent generalized Cuddeford systems is introduced and the inversion formula obtained, together with the associated consistency conditions.

3.1 Cuddeford systems

An interesting generalization of OM and constant anisotropy systems was proposed by Cuddeford (1991; see also Ciotti 2000, chapter 10) assuming

$$f = J^{2\alpha} h(Q), \quad (14)$$

where $\alpha > -1$ is a real number and Q is defined as in equation (1): isotropic models then correspond to $\alpha = 0$ and $r_a \rightarrow \infty$. Equation (14) can be used to describe both the OM models (for $\alpha = 0$) and the constant anisotropy models (for $r_a \rightarrow \infty$). In particular, the anisotropy parameter takes now the simple form

$$\beta(r) = \frac{r^2 - \alpha r_a^2}{r^2 + r_a^2} \quad (15)$$

(see equations A1–A5). Therefore, when $\alpha > 0$ the anisotropy is tangential in the inner regions where $r < \sqrt{\alpha} r_a$, and radial for $r > \sqrt{\alpha} r_a$; in the limit $\alpha \rightarrow \infty$, the orbital structure is fully tangentially anisotropic (i.e. $\beta \rightarrow -\infty$). Instead, when $-1 < \alpha < 0$ the models are radially anisotropic everywhere, independent of the value of r_a ;

moreover, in the limit $\alpha \rightarrow -1$ equation (15) gives $\beta \rightarrow 1$, so that the velocity anisotropy is completely radial.

The DF of a Cuddeford system and its spatial density are related as

$$\rho(r) = 2\sqrt{8\pi} A(r, \alpha) \int_0^{\Psi_T} (\Psi_T - Q)^{\alpha+1/2} h(Q) dQ, \quad (16)$$

where

$$A(r, \alpha) = 2^{\alpha-1} \sqrt{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)} \frac{r^{2\alpha}}{(1+r^2/r_a^2)^{\alpha+1}} \quad (17)$$

and $\Gamma(x) = (x-1)!$ is the gamma function. As expected, equation (2) is re-obtained for $\alpha = 0$, while the convergence of the angular part of the integral over the velocity space requires $\alpha > -1$. In analogy with the discussion in Section 2, the augmented density

$$\begin{aligned} \varrho(r) &\equiv \frac{\rho}{A(r, \alpha)} \\ &= \frac{2^{1-\alpha}}{\sqrt{\pi}} \frac{\Gamma(\alpha+3/2)}{\Gamma(\alpha+1)} \left(1 + \frac{r^2}{r_a^2}\right)^{\alpha+1} \frac{\rho(r)}{r^{2\alpha}} \end{aligned} \quad (18)$$

is introduced, and a simple inversion formula, similar to equation (7), permits to recover the DF from the density profile. In fact, after

$$m = \text{int} \left(\alpha + \frac{1}{2} \right) + 1 \quad (19)$$

differentiations² with respect to Ψ_T , equation (16) can be Abel inverted (Cuddeford 1991). In practice, one must perform enough differentiations as to produce a negative exponent (> -1) in the power-law kernel of integral (16).

When $\alpha > -1$ (i.e. $m \geq 0$) but α is not half-integer,

$$\begin{aligned} h(Q) &= \frac{(-1)^{m+1} \cos \alpha\pi}{2\sqrt{8\pi}^2} \frac{\Gamma(\alpha+3/2-m)}{\Gamma(\alpha+3/2)} \\ &\times \frac{d}{dQ} \int_0^Q \frac{d^m \varrho}{d\Psi_T^m} \frac{d\Psi_T}{(Q-\Psi_T)^{\alpha+3/2-m}} \\ &= \frac{(-1)^{m+1} \cos \alpha\pi}{2\sqrt{8\pi}^2} \frac{\Gamma(\alpha+3/2-m)}{\Gamma(\alpha+3/2)} \\ &\times \int_0^{\Psi_T} \frac{d^{m+1} \varrho}{d\Psi_T^{m+1}} \frac{d\Psi_T}{(Q-\Psi_T)^{\alpha+3/2-m}}, \end{aligned} \quad (20)$$

where the last identity holds for untruncated systems with finite total mass and the OM inversion formula (7) is reobtained for $\alpha = 0$.

When α is half-integer, i.e. $\alpha = m - 3/2$ and $m = 1, 2, \dots$, the solution of the Volterra equation (16) is given by

$$h(Q) = \frac{1}{2\sqrt{8\pi}(m-1)!} \left[\frac{d^m \varrho}{d\Psi_T^m} \right]_{\Psi_T=Q}, \quad (21)$$

and the DF is recovered analytically avoiding integration.³

3.2 The consistency criteria and the density slope–anisotropy inequality

As we now show, the inversion formulae (20) and (21) still hold for the more general case of multicomponent, generalized Cuddeford systems, in which the DF associated with *each* density component

² $\text{int}(x)$ means the largest integer $\leq x$. For example, $\text{int}(1/2) = 0$ and so $m = 1$ for OM models.

³ In equation (30) of Cuddeford (1991), the $(m-1)!$ at the denominator is missing. See also equations (49) and (51) of Baes & Dejonghe (2002).

is made by the sum of an arbitrary number of Cuddeford DFs with arbitrary positive weights w_i and possibly different anisotropy radii r_{ai} (but same h function and angular momentum exponent), i.e.

$$f = J^{2\alpha} \sum_i w_i h(Q_i), \quad Q_i = \mathcal{E} - \frac{J^2}{2r_{ai}^2}. \quad (22)$$

The different density components of a multicomponent generalized Cuddeford system will have, in general, a different value of α and a different function $h(Q)$. As should be clear, all the results presented in Sections 2 and 3.1 hold as special cases of the following treatment.

Of course, the orbital anisotropy distribution characteristic of DF (22) is *not* a Cuddeford one: as shown in the Appendix, the anisotropy function $\beta(r)$ of each density component is given by

$$\beta(r) = 1 - (\alpha + 1) \frac{\sum_i w_i / (1 + r^2/r_{ai}^2)^{\alpha+2}}{\sum_i w_i / (1 + r^2/r_{ai}^2)^{\alpha+1}}. \quad (23)$$

Quite general anisotropy profiles can be obtained by specific choices of the weights w_i , the anisotropy radii r_{ai} and the exponent α . However, near the centre $\beta(r) \sim -\alpha$, and $\beta(r) \sim 1$ for $r \rightarrow \infty$, independent of the specific values of w_i and r_{ai} .

We now show that an Abel inversion formula identical to equations (20)–(21) can be found for a DF of the family (22). In fact, it is immediate to verify that equation (16) still holds, where now the radial function is

$$A(r, \alpha) = \frac{\sqrt{\pi} \Gamma(\alpha + 1)}{2 \Gamma(\alpha + 3/2)} \sum_i \frac{w_i r^{2\alpha}}{(1 + r^2/r_{ai}^2)^{\alpha+1}}, \quad (24)$$

and so, once the new augmented density $\varrho = \rho/A$ is defined, the function h in equation (22) can in principle be recovered. Therefore, it is obvious that the same arguments used to derive the necessary and sufficient conditions for consistency of OM models can be repeated also for each density component of multicomponent generalized Cuddeford systems. However, as m differentiations with respect to Ψ_T must be performed on the integral (16) before the inversion, we now obtain m NCs and a sufficient condition. Surprisingly, as in the case of OM models, we found that the first of the NCs for consistency can be rewritten as the density slope–anisotropy theorem which must hold at every radius. These results are summarized in the following.

Theorem: each density component in a consistent multicomponent generalized Cuddeford system with α not half-integer obeys m NCs (NC $_k$):

$$\frac{d^k \varrho}{d\Psi_T^k} \geq 0, \quad k = 1, 2, \dots, m, \quad (25)$$

where m is given by equation (19). In particular, the NC $_1$ can be rewritten as the density slope–anisotropy inequality

$$\gamma(r) \geq 2\beta(r), \quad \forall r. \quad (26)$$

Moreover, a sufficient condition for the non-negativity of the DF of each component is

$$\frac{d^{m+1} \varrho}{d\Psi_T^{m+1}} \geq 0. \quad (27)$$

Proof: a proof of the m NCs (25) is obtained by repeated differentiation of the augmented density ϱ [see equations 16–18, where now $A(r, \alpha)$ is given by equation 24] with respect to the total potential Ψ_T , and by the assumption that $h(Q)$ is a positive function. The sufficient condition (27) is derived just by imposing the positivity

of the integrand in the second of identities (20). Finally, we refer to the Appendix for a proof of inequality (26). Of course, in the special cases of $\alpha = m - 3/2$ and $m = 1, 2, \dots$, equation (21) provides, in addition to the $m - 1$ NCs (25), the necessary and sufficient condition for consistency of the specific component, i.e.

$$\frac{d^m \varrho}{d\Psi_T^m} \geq 0. \quad (28)$$

As expected, the NC and the WSC derived in CP92 are reobtained as special cases of the new theorem for $\alpha = 0$ (i.e. $m = 1$). In applications, as those that will be presented in Section 4, it is useful to express equations (25) and (27) in terms of the radius. From the relation $d\Psi_T/dr = -GM_T(r)/r^2$, where $M_T(r)$ is the total mass enclosed by the radius r , the NC $_1$ can be expressed as

$$\frac{d\varrho}{dr} \leq 0, \quad (29)$$

which is a second alternative formulation of the density slope–anisotropy theorem in addition to equation (26). Following the same approach, the NC $_2$ can be also expressed as

$$\frac{d}{dr} \left[\frac{r^2}{M_T(r)} \frac{d\varrho}{dr} \right] \geq 0, \quad (30)$$

and so on, with the sign of the NC $_k$ inequality alternating with increasing k . Finally, note that NC $_1$ is the sole condition in which only the augmented density profile of the specific density component appears, while in the higher order NC $_k$ the total mass profile $M_T(r)$ is also involved.

4 SOME ILLUSTRATIVE CASES

In the previous section, we derived the family of necessary and sufficient conditions for phase-space consistency of each density component of generalized Cuddeford systems, and showed that the density slope–anisotropy inequality holds at every radius.

We now present a simple application of the new consistency criteria, and address two natural questions concerning phase-space consistency of Cuddeford systems. The first is related to the fact that for $\alpha \geq 1/2$ we have, at variance with the OM case ($\alpha = 0$), more than one NC for the non-negativity of the DF. Which NC is stronger? Or, more quantitatively, which of the NC $_k$ gives a consistency limit closer to the true one (that would be derived from the DF)? The second question is: for a given density profile, what is the effect of tangential anisotropy on consistency? Will the minimum anisotropy radius increase or decrease at increasing α , i.e. at increasing tangential anisotropy? The set of NCs and the dependence of their number on α through equation (19) suggest a simple approach to address the two issues above. Consider an assigned density profile, representing a component in a multicomponent generalized Cuddeford system: what is the behaviour of the consistency region in parameter space at increasing α ? At increasing α , the number of NCs increases: of course each additional NC can only reduce the consistency region in the parameter space. In addition, when α increases so that m given by equation (19) increases by 1, the former sufficient condition NC $_{m+1}$ becomes the last of the NCs for the new model.

We now illustrate the procedure, and discuss the two questions presented above, by investigating the phase-space consistency of the widely used γ -models (Dehnen 1993; Tremaine et al. 1994; see equation 31). We do this in the most simplified form, i.e. in the case of a one-component Cuddeford system; in other words, in equation (22) we restrict to $i = 1$. The detailed study of $\gamma = 0, 1$ (Hernquist 1990) and $\gamma = 2$ (Jaffe 1983) models will also allow us to explore the combined effect of the inner density slope

and of tangential anisotropy (Section 4.1), while the additional role played by the external density slope will be discussed in Section 4.2 by using one-component $n-\gamma$ models with Cuddeford anisotropy (see equation 34). We recall that a consistency analysis of OM anisotropic $n-\gamma$ models was done in CM09.

4.1 The one-component γ -models

We start by considering the general γ -model, whose dimensionless density profile and mass enclosed inside radius r are given by

$$\rho(r) = \frac{1}{s^\gamma(1+s)^{4-\gamma}}, \quad (31)$$

$$M(r) = \left(\frac{s}{1+s}\right)^{3-\gamma}, \quad 0 \leq \gamma < 3, \quad (32)$$

where $s \equiv r/r_c$ is the radius normalized to the ‘core’ radius r_c . It is trivial to show that the consistency properties of one-component models are independent of the mass and density normalization scales.

In the following, we will study the NC_k functions by using their radial formulation (equations 29–30 with the augmented density of equation 18). Indeed, in common situations the elimination of the radius from the density profile in favour of the gravitational potential, needed to evaluate equation (25), is not feasible. For this reason, we prefer to study the consistency conditions by using their radial expressions, as this procedure is always viable, for whatever density profile expressed as a function of radius.

We begin by noting that from AE06 theorem we already know that $\alpha \geq -\gamma/2$ is required at the centre of the density distribution (31); this condition must be combined with $\alpha > -1$ (see Section 3.1). Once the appropriate augmented density is defined, the radial NC_1 for Cuddeford anisotropic γ -models read with equation (18) reduces to

$$s_a^2[2s(2+\alpha) + 2\alpha + \gamma] + s^2(2s + \gamma - 2) \geq 0, \quad \forall s, \quad (33)$$

thus establishing a relation between α and the normalized anisotropy radius $s_a \equiv r_a/r_c$.

As expected, for $s = 0$ the inequality above reduces to the AE06 limitation. However, as the condition (33) must hold over the entire radial range, we can now derive limitations on the minimum allowed anisotropy radius s_a as a function of γ and α . The general formula is simple but here we prefer to focus on the special cases of $\gamma = 0, 1$ and 2 . The NC_1 is represented by a solid curve in Figs 1–3, respectively, for the $\gamma = 0, 1$ and 2 models; of course, while α is restricted to positive values when considering the $\gamma = 0$ case, the α axis begins at $\alpha = -0.5$ for $\gamma = 1$ models, and finally the AE06 limitation in the $\gamma = 2$ case is $\alpha > -1$ (coincident with the value required by convergence of the integral in equation 16).

In the three figures, all points below the solid NC_1 curve correspond to unphysical models, while points above may represent consistent models. The solid dots are the true lower limits on s_a determined by direct inspection of the DF for representative values of α : the $\gamma = 1$ case was already given by Baes & Dejonghe (2002). Note that for the Jaffe model (Fig. 3) the NC_1 actually coincides with the abscissae axis, i.e. it is satisfied for all values of $\alpha > -1$, independent of the value of s_a . In the case of $\gamma = 1$ and 2 models, when $\alpha = -1/2$ equation (21) provides the DF, and so the true critical anisotropy radius s_a can easily be determined (black dots in Fig. 2 and 3). Of course, these values coincide with those obtained from the NC_1 for $\alpha = -1/2$, as should be clear from the discussion in Section 3. The solid dots at $\alpha = 0$ represent instead the

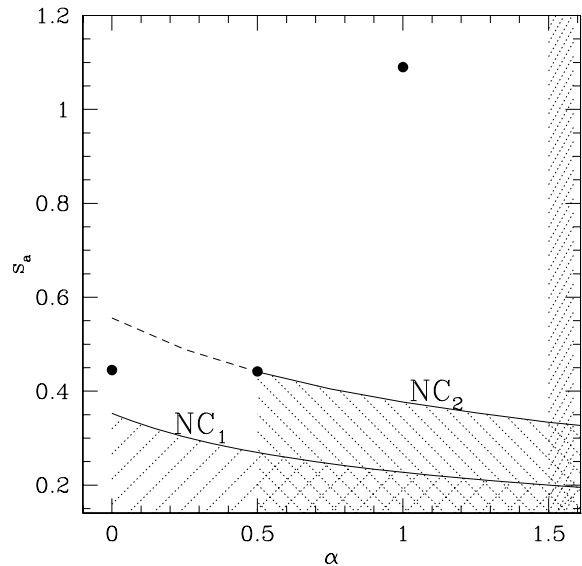


Figure 1. Consistency limits on the normalized anisotropy radius $s_a = r_a/r_c$ for the one-component $\gamma = 0$ model with Cuddeford anisotropy. The solid curves mark the limits imposed by NC_1 and NC_2 : models with the pair (α, s_a) in the shaded regions are certainly inconsistent; models above the dashed curve [where NC_2 is the sufficient condition (27)] are certainly consistent. Solid dots are the true lower limits for s_a derived from the DF. Recall that $\alpha = 0$ refers to the OM model. No consistent models exist for $\alpha \geq 3/2$.

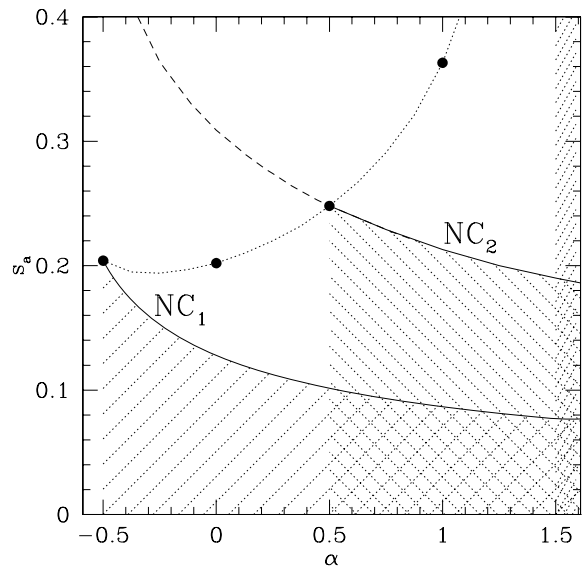


Figure 2. Consistency limits on the normalized anisotropy radius $s_a = r_a/r_c$ for the one-component Hernquist ($\gamma = 1$) model with Cuddeford anisotropy. Different curves have the same meaning as in Fig. 1. The dotted line connecting the solid dots has been obtained from table 1 in Baes & Dejonghe (2002).

DF-derived lower limit for the minimum anisotropy radius for the corresponding OM models (see e.g. C96). As the NC_1 for $-0.5 < \alpha < 0.5$ is just an NC for consistency, while the NC_2 provides a *sufficient* condition for consistency (i.e. all points above the NC_2 dashed curve correspond to consistent models), it is not surprising that for all the three density models the DF-derived limit on the anisotropy radius in the OM case is contained in the region delimited by the NC_1 and the NC_2 (see C99, table 1).

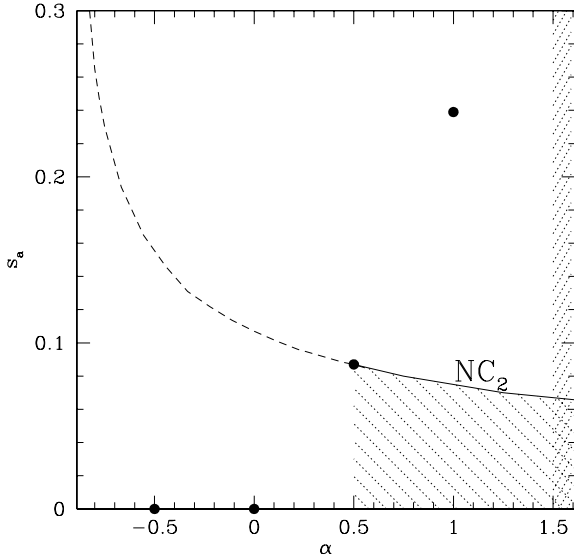


Figure 3. Consistency limits on the normalized anisotropy radius $s_a = r_a/r_c$ for the one-component Jaffe ($\gamma = 2$) model with Cuddeford anisotropy. Different curves have the same meaning as in Fig. 1. Note that the NC_1 coincides with the x -axis.

As we increase α , when we reach the value $\alpha = 1/2$ the NC_2 function becomes the model DF, and so the DF-derived lower limit, represented with a black dot, coincides again with the critical curve.

For $1/2 < \alpha < 3/2$, the NC_2 becomes a new NC, and therefore all points in the shaded area below the solid NC_2 curves in the three figures correspond to unphysical models. Note how the NC_2 provides more stringent limits than the NC_1 . Consistently with the nature of the NC_2 , the black dots representing the limits on s_a derived from the DF for $\alpha = 1$ lie above the NC_2 curve. Of course, in this range of values of α the NC_3 is the sufficient condition for phase-space consistency. However, an asymptotic expansion of the NC_3 for $s \rightarrow \infty$ easily shows that this condition is violated, independent of the value of s_a and α . This fact poses no problem in the range $1/2 < \alpha < 3/2$, as NC_3 is a sufficient condition there, but as soon as α becomes larger than $3/2$ the NC_3 becomes necessary, and the whole family of γ -models with Cuddeford anisotropy becomes inconsistent. We note that the $\alpha = 3/2$ limitation was already determined by Baes & Dejonghe (2002) for Hernquist models with Cuddeford anisotropy. Quite surprisingly, by using the NC_3 we found that the limitation $\alpha < 3/2$ holds for the entire family of γ -models, no matter which value of γ is considered. The reason is due to the fact that the *external* density slope of γ -models is 4, independent of the value of γ . Thus, while the lower limit on α is due to the central density slope, the external density slope limits the amount of tangential anisotropy that can be supported by the models. This indication is very interesting, because it means that the external regions (where anisotropy is almost completely radial, see equation 15) are able to affect the inner dynamics. We will discuss such issue in Section 4.2.

As a final remark, we note that a comparison of Figs 1–3 confirms qualitatively the trend already found in Carollo, de Zeeuw & van der Marel (1995), C96 and C99 for one-component OM models. In practice, at fixed α the minimum anisotropy radius increases at decreasing inner density slope γ , i.e. centrally flatter density profiles are less able to sustain radial anisotropy than steeper density profiles, even in presence of a central tangential anisotropy. This is shown by the smaller and smaller shaded areas at fixed α and increasing γ , and by the corresponding smaller exact values of the minimum s_a indicated by the solid dots.

4.2 The effect of the external density slope

As we have seen, no Cuddeford anisotropic γ -model exists for $\alpha \geq 3/2$, and this is independent of the value of γ . The fact that the critical upper limit of α is independent of γ is a clear indication of the importance of the slope of the outer density profile on the central anisotropy. However, being the limit imposed on α , it implies that the central regions are the ones affected. A hint to understand this phenomenon is given by inspection of the three figures: in fact, note how the consistency region in the (α, s_a) space reduces at increasing α , in the sense that for increasing α the minimum value of s_a increases. This means that when the central regions are forced to be more and more tangentially anisotropic, the external regions (where s_a determines the amount of radial anisotropy) must be more and more isotropic. Therefore, we conclude that the origin of the inconsistency at high α is a combination of the forced tangential anisotropy and the radial orbits arriving from the external regions of the system.

To better understand this behaviour, we now consider the one-component, Cuddeford anisotropic n - γ models, whose normalized density profile is given by

$$\rho = \frac{1}{s^\gamma(1+s)^{n-\gamma}}, \quad 0 \leq \gamma < 3, \quad n > 3 \quad (34)$$

(see CM09). If our previous argument is correct then the maximum value of α should increase at increasing n , as less and less mass is contained outside the core radius at increasing n , so that less and less radial orbits can affect the inner regions. Unfortunately, for generic (non integer) values of n the mass contained within r cannot be expressed in terms of elementary functions. However, it is possible to perform an asymptotic analysis at large radii of the NC_k (with some care, as differentiation of asymptotic relations is usually not permitted; e.g. see Bender & Orszag 1978). The appropriate way to perform the analysis in this case is to use the NC_k formulated in terms of the potential (equation 25), and to adopt the asymptotic expansion for the relative potential $\Psi = 1/s + O(1/s^2)$. Following this approach, it can be proved that, independent of the value of the inner density slope γ , the critical value of α increases with n : for example, when $n = 5$ it is required that $\alpha < 5/2$, when $n = 6\alpha < 7/2$ and so on. This confirms the previous conjecture.

5 DISCUSSION AND CONCLUSIONS

In a natural extension of previous investigations (CP92, C96, C99, AE06, CM09), we searched for phase-space consistency criteria for multicomponent spherical systems. We found that inversion formulae and necessary and sufficient conditions for consistency can actually be derived for multicomponent generalized Cuddeford systems. Such systems contain as very special cases OM, constant anisotropy and Cuddeford models. The main results of our study can be summarized as follows.

(i) New phase-space consistency criteria, i.e. necessary and sufficient conditions for the DF non-negativity, are derived for multicomponent, generalized Cuddeford systems. At variance with the simpler case of OM models, the presence of tangential anisotropy leads to a *family* of NCs, that can be written as simple inequalities involving repeated differentiations of the augmented density expressed as a function of the total potential.

(ii) It is shown that the first of the NCs for consistency can be reformulated as the density slope–anisotropy theorem, which therefore is proved to hold not only at the centre *but also at all radii for each density component of multicomponent generalized Cuddeford models*.

(iii) The first NC is the only condition independent of the other density components of the model. All the other (more stringent) conditions depend on the total density distribution of the model.

(iv) All the conditions can be reformulated in term of the radius, so that they can be tested also for models in which the total potential cannot be expressed by using elementary functions or when the radius cannot be eliminated in favour of the potential.

(v) The new phase-space consistency criteria are applied to one-component γ -models with Cuddeford anisotropy. It is found that for increasing tangential anisotropy in the central regions the minimum anisotropy radius for consistency increases, i.e. the external regions must be less and less radially anisotropic. No consistent γ -models exist for $\alpha \geq 3/2$, independent of the value of the central density slope γ . Baes & Dejonghe (2002) already found this limitation by direct inspection of the DF of Hernquist models with Cuddeford anisotropy.

(vi) To investigate the combined effect of the outer radial and inner tangential anisotropy, we performed an asymptotic analysis of one-component n - γ models with Cuddeford anisotropy. We found that a steepening of the external density slope allows larger values of the central tangential anisotropy, independent of the value of the central density slope γ , thus confirming the hypothesis of a dynamical interplay between the two regions of the models and supporting the interpretation that Baes & Dejonghe (2002) proposed for Hernquist models.

We note that one of the major results of this study seems to be the generality of the density slope–anisotropy relation $\gamma(r) \geq 2\beta(r)$. It is natural to ask whether such density slope–anisotropy relation is even more general, i.e. it is an inequality necessarily obeyed by generic spherically symmetric, two-integrals systems with positive DF. At this stage, we do not have a proof of this conjecture, but we are not aware of any counterexample. Actually, we have additional evidences supporting this conjecture: for example, Michele Trenti kindly provided us with a large set of numerically computed f_v models (Bertin & Trenti 2003), and all of them, without exception, satisfy the inequality $\gamma(r) \geq 2\beta(r)$ at all radii (see also fig. 7 of Hansen 2008). Moreover, it is trivial to show that spherical systems in which the density can be written as $\rho = A(r)f(\Psi)$, with f monotonically increasing function of Ψ , all obey to $\gamma(r) \geq 2\beta(r)$ when supported by a positive DF (see also the comment after equation A9). We stress that these models do not belong to the family of generalized Cuddeford systems. Other distributions of orbital anisotropy that are not of the Cuddeford family (even though they could be approximated by specific choices of generalized Cuddeford distributions) have been reported by Mamon & Lokas (2005), Wojtak et al. (2008, who went further to show that also the DF was not of the OM or Cuddeford forms) and Ascasibar & Gottlöber (2008) from the analysis of haloes in cosmological simulations, or proposed in terms of specific DFs (e.g. Gerhard 1991; Louis 1993; Cuddeford & Louis 1995): it would be interesting to check the $\gamma(r) \geq 2\beta(r)$ inequality in these systems. In any case, we note that numerical simulations are known to produce correlations between β and γ (e.g. see Hansen & Moore 2006; Mamon et al. 2006). We finally conclude by noting that, if the inequality $\gamma(r) \geq 2\beta(r)$ is universal (for spherical systems), then the so-called mass–anisotropy degeneracy could be less severe than what is sometimes feared, as orbital anisotropy would be in some sense controlled by the local density slope of the stellar distribution in galaxies (in the inner regions where $\gamma \leq 2$). This could be an important constraint in observational works.

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APPENDIX A: THE DENSITY SLOPE–ANISOTROPY INEQUALITY FOR GENERALIZED CUDDEFORD SYSTEMS

The radial and tangential velocity dispersion profiles of each density component of a multicomponent generalized Cuddeford system are

given by

$$\begin{aligned} \rho\sigma_r^2 &= \int f v_r^2 d^3v \\ &= 4\pi B(r, \alpha) \int_0^{\Psi_T} [2(\Psi_T - Q)]^{\alpha+3/2} h(Q) dQ, \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} \rho\sigma_i^2 &= \int f v_i^2 d^3v \\ &= 4\pi C(r, \alpha) \int_0^{\Psi_T} [2(\Psi_T - Q)]^{\alpha+3/2} h(Q) dQ, \end{aligned} \quad (\text{A2})$$

where

$$B(r, \alpha) \equiv \frac{\sqrt{\pi}}{4} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 5/2)} \sum_i \frac{w_i r^{2\alpha}}{(1 + r^2/r_{ai}^2)^{\alpha+1}}, \quad (\text{A3})$$

$$C(r, \alpha) \equiv \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha + 5/2)} \sum_i \frac{w_i r^{2\alpha}}{(1 + r^2/r_{ai}^2)^{\alpha+2}}. \quad (\text{A4})$$

Then, from equation (4) one has

$$\beta(r) = 1 - \frac{C(r, \alpha)}{2B(r, \alpha)}, \quad (\text{A5})$$

and simple algebra proves equation (23).

We now show that the inequality $\gamma(r) \geq 2\beta(r)$ holds at all radii in each density component of consistent generalized Cuddeford systems. First, we relate the logarithmic density slope $\gamma(r) \equiv -d \ln \rho / d \ln r$ to the NC_1 (29) as follows:

$$0 \geq \frac{dQ}{dr} = \frac{d}{dr} \frac{\rho}{A} = \frac{\rho}{rA} \frac{d \ln(\rho/A)}{d \ln r}, \quad (\text{A6})$$

so that the NC_1 can be simply rewritten as

$$\gamma(r) \geq -\frac{d \ln A}{d \ln r}. \quad (\text{A7})$$

In other words, all consistent generalized Cuddeford systems satisfy equation (A7) at each radius. Now it is easy to verify that the functions A , B and C , given in equations (24), (A3) and (A4), satisfy the identity

$$-\frac{d \ln A}{d \ln r} = 2 \left[1 - \frac{C(r, \alpha)}{2B(r, \alpha)} \right] = 2\beta \quad (\text{A8})$$

for arbitrary α , w_i and r_{ai} , so that equations (A7) and (A8) show that the inequality $\gamma(r) \geq 2\beta(r)$ is just another way to express the NC_1 . Identity (A8) can be proved by elementary algebra:

$$\begin{aligned} -\frac{d \ln A}{d \ln r} &= -2\alpha \\ &+ 2(\alpha + 1) \sum_i \frac{w_i r^2 + w_i r_{ai}^2 - w_i r_{ai}^2}{r_{ai}^2 (1 + r^2/r_{ai}^2)^{\alpha+2}} \left[\sum_i \frac{w_i}{(1 + r^2/r_{ai}^2)^{\alpha+1}} \right]^{-1}, \end{aligned} \quad (\text{A9})$$

where in each term of the sum we added and subtracted $w_i r_{ai}^2$. Simplification and comparison with equation (23) conclude the proof.

We note that identity (A8) is actually a special case of a more general result reported in Baes & Dejonghe (2002) and Baes & van Hese (2007), which holds for *all* spherical systems whose DF, after integration over velocity space, leads to the factorization $\rho(r) = A(r)f(\Psi)$; for such systems (that also include our generalized Cuddeford systems), it can be proved that $2\beta(r) = -d \ln A / d \ln r$.

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