

A fast and accurate method to compute the mass return from multiple stellar populations

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Accepted 2014 February 26. Received 2014 January 22; in original form 2013 December 2

ABSTRACT

The mass returned to the ambient medium by aging stellar populations over cosmological times sums up to a significant fraction (20–30 per cent or more) of their initial mass. This continuous mass injection plays a fundamental role in phenomena, such as galaxy formation and evolution, fuelling of supermassive black holes in galaxies and the consequent (negative and positive) feedback phenomena, and the origin of multiple stellar populations in globular clusters. In numerical simulations, the calculation of the mass return can be time consuming, since it requires at each time step the evaluation of a convolution integral over the whole star formation history, so the computational time increases quadratically with the number of time steps. The situation can be especially critical in hydrodynamical simulations, where different grid points are characterized by different star formation histories, and the gas cooling and heating times are shorter by orders of magnitude than the characteristic stellar lifetimes. In this paper, we present a fast and accurate method to compute the mass return from stellar populations undergoing arbitrarily complicated star formation histories. At each time step the mass return is calculated from its value at the previous time, and the star formation rate over the last time step only. Therefore, in the new scheme there is no need to store the whole star formation history, and the computational time increases linearly with the number of time steps.

Key words: methods: numerical – galaxies: abundances – galaxies: ISM – Galaxies: stellar content.

1 INTRODUCTION

For a simple stellar population (SSP), the mass return rate from the aging stars to the ambient medium depends on the relation between the initial mass and the remnant mass of each star, and the initial mass function (hereafter IMF; e.g. Tinsley 1980; Matteucci & Greggio 1986; Tosi 1988; Ciotti et al. 1991; Maraston 1998). In general, the mass return of an SSP represents a non-negligible fraction of its initial mass, ranging from 20 to 30 per cent for standard choices of the IMF (such as a Scalo 1986; Kroupa, Tout & Gilmore 1993; Chabrier 2003; e.g. Pellegrini 2012).

In stellar systems, this source of fresh gas is present independently of random phenomena, such as galaxy merging; therefore, the mass return of stellar populations plays a major role in determining the chemical composition and the baryonic mass budget of the host systems. For example, the gas recycled by the aging stellar population is the main mass source for gas flows in early-type galaxies, such as cooling flows and galactic winds (for general reviews see, e.g. Mathews & Brighenti 2003; Pellegrini 2012), for accretion of

supermassive black holes (SMBHs) at the centre of spheroids (e.g. Norman & Scoville 1988; Padovani & Matteucci 1993; Tabor & Binney 1993; Ciotti & Ostriker 1997; Shcherbakov et al. 2014, see also Ciotti & Ostriker 2012, and references therein), and the consequent *negative and positive feedback* (e.g. Ciotti & Ostriker 2007; Ishibashi & Fabian 2012; Zubovas et al. 2013). Another case where the mass returned from the evolving stars seems to play a fundamental role is the origin of multiple stellar populations in globular clusters (see, e.g. Piotto et al. 2007; Renzini 2008; D’Ercole et al. 2012).

Of course, real stellar systems are made by multiple stellar populations, i.e. by a collection of SSPs assembled at different epochs and with different metallicities, so that the mass return rate is a function of their star formation history (SFH). In this case, an accurate calculation of the mass-loss rate requires keeping track of the age and metallicity of each SSP and, depending on the desired time resolution, its computation can be expensive in terms of computer time. In fact, the function describing the mass return of a stellar population is in general linked to its SFH through a convolution integral over time. Therefore, not only the numerical evaluations needed for its computation increase quadratically with the number of time steps, but also the whole SFH must be stored in the computer memory. The

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situation becomes extremely time- and memory-consuming in the case of hydrodynamical simulations, where each cell of the numerical grid in principle hosts a different SFH, and where the number of time steps can be of the order of millions or more for simulations spanning a Hubble time, due to heating, cooling and Courant times that can be orders of magnitude shorter than the characteristic stellar lifetimes.

Similar (but less severe) problems affect also semi-analytic models (SAMs) of galaxy formation (e.g. Baugh 2006). In such models, the evolution of the baryonic matter is driven by the evolution of the dark matter haloes, and the most massive galaxies form by the progressive coalescence of a large number of progenitors. Thanks to the simplified treatment of the various physical processes involved, SAMs allow to explore the parameter space at a reasonable computational cost. However, in order to compute accurately the mass return rate with an acceptable time resolution ($\simeq 10$ Myr or less) over a Hubble time, the most massive galaxies require to store the SFHs of several thousand progenitors (and this for an array of $\simeq 10^2$ elements if one is interested in chemical evolution). To reduce the computational cost, sometimes the mass return rate is computed by means of the *instantaneous recycling approximation* (IRA), i.e. assuming that massive stars contribute instantaneously to the mass return and to the chemical enrichment of the interstellar medium (ISM), whereas the contribution of low- and intermediate-mass stars is neglected at any epoch (e.g. Starkenburg et al. 2013). This approach has considerable limitations, since at short times after a burst of star formation it can lead to a severe overestimation of the mass return rate of the stellar populations (e.g. Matteucci 2001). Note that the IRA is used also in hydrodynamical simulations, for instance when studying the evolution of star-forming molecular clouds (where the required time resolution is set by the cooling time, of the order of 10^5 yr, e.g. Krumholz 2011).

In order to overcome these problems and to compute accurately the mass reprocessing from evolving stellar populations, in this paper, we present a fast and very accurate method which takes into account the stellar lifetimes, at significantly reduced computational cost with respect to the direct integration of the convolution integral. The method bypasses the need of storing the SFH, as it uses information relative to the previous time step only, thus reducing the number of evaluations of the convolution integral from quadratic to linear. The new scheme can be easily implemented in SAMs and in hydrodynamical codes, with significant gain in accuracy and speed over standard methods. The basic idea extends a scheme already adopted in numerical simulations to describe the delayed accretion of gas from the accretion disc to the SMBH at the centre of stellar spheroids, where the mass flow on the accretion disc is determined by the solution of the hydrodynamical equations for the ISM (e.g. Ciotti & Ostriker 2007). A simplified version of this method has been implemented in Lusso & Ciotti (2011) to describe the time evolution of the Type Ia supernova (SNIa) rate in galaxy formation models with AGN feedback. In fact, the new method here presented can be applied not only to the computation of the mass return rate from stellar populations, but also to other astrophysical studies where time-dependent convolution integrals with known kernels must be computed (as in the case mentioned above of SNIa, e.g. Greggio 2005 for a full discussion).

The paper is organized as follows: In Section 2, we introduce the basic equations for the mass return rate and the formalism underlying the new method. In Section 3, illustrative results are presented, and finally, in Section 4, we draw our conclusions. Mathematical details behind the method are given in the appendix.

2 THE NUMERICAL PROBLEM

We start by considering an SSP of total mass M_* and IMF described by the function $\Psi(M) = M_*\phi(M)$; from now on all stellar masses are in solar mass units, M_\odot . The *normalized* IMF $\phi(M)$ is defined so that

$$\int_{M_{\text{inf}}}^{M_{\text{sup}}} M \phi(M) dM = 1, \quad (1)$$

where M_{inf} and M_{sup} are the minimum and the maximum stellar mass in the population, respectively.

As well known (e.g. Ciotti et al. 1991; Pellegrini 2012), for an SSP assembled at $t = 0$, the normalized mass return rate r_{SSP} can be written as

$$r_{\text{SSP}}(t) = \phi[M_{\text{TO}}(t)] \times M_{\text{ej}}[M_{\text{TO}}(t)] \times |\dot{M}_{\text{TO}}(t)|, \quad (2)$$

where $M_{\text{TO}}(t)$ is the mass of the stars entering the turn-off phase at time t , and $M_{\text{ej}}(M)$ is the total mass ejected by a star of initial mass M in the post main sequence evolutionary phases. By construction, the mass return rate of the considered SSP is given by $R_{\text{SSP}} = M_* \times r_{\text{SSP}}$.

For an SFH characterized by a star formation rate $\psi(t)$, and a time- and metallicity-independent IMF, the mass return rate at the time t is given by

$$R(t) = \int_0^t \psi(\tau) r_{\text{SSP}}(t - \tau) d\tau. \quad (3)$$

Note that the lower limit of integration is arbitrary, being related to the beginning of star formation: for example, in this case, we assume without loss of generality that $\psi = 0$ for $t < 0$. On the contrary, the upper limit of integration is physically required by the obvious fact that mass return from a stellar population cannot take place before its formation.

As pointed out in the Introduction, the integral above, when evaluated by direct sum, scales quadratically with the number of time steps adopted to discretize the time t , so its computation can be extremely time-consuming. We will now show how to evaluate equation (3) as a *linear* function of the number of time steps, and without the need of storing the whole evolution of $\psi(t)$. We will refer to the rate as obtained by integrating equation (3) with standard methods as the *exact rate*, while the alternative scheme here presented will be called the *new method*.

In the formalism of equation (3), the mass return rate in the IRA is written as

$$R_{\text{IRA}}(t) = \psi(t) \int_0^\infty r_{\text{SSP}}(\tau) d\tau : \quad (4)$$

in practice, the IRA mass return rate is just given by the instantaneous star formation rate times the total mass fraction released by the SSP. Of course, the IRA overcomes all the problems of computational times and memory storage posed by the standard evaluation of equation (3), but it can be properly used only when *all* the time-scales of the problem under consideration are much longer than the lifetimes of the typical stars producing the bulk of the mass return in the SSP.

2.1 The new method

The idea behind the new method is to substitute the exact kernel r_{SSP} in equation (3) with a sum of functions with special mathematical properties. We begin by illustrating the method for the case of a sum of pure exponentials. Suppose that for a given SSP formed at

$t = 0$ the associated $r_{\text{SSP}}(t)$ can be represented with high accuracy as

$$r_{\text{SSP}}(t) = \sum_{i=1}^k \alpha_i e^{-\beta_i t}, \quad (5)$$

where the parameters α_i and β_i are obtained by fitting the exact function given in equation (2); note that α_i and β_i are in units of inverse of time (e.g. Gyr^{-1}).

It follows that equation (3) can be written as

$$R(t) = \sum_{i=1}^k \alpha_i I_i^{(0)}(t), \quad (6)$$

where

$$I_i^{(0)}(t) = \int_0^t \psi(\tau) e^{-\beta_i(t-\tau)} d\tau. \quad (7)$$

The meaning of the superscript ‘(0)’ will become clear in the following. For simplicity we drop the subscript index i , as we now derive the generic expression for the family $I^{(0)}$, and the restoration of subscript i in the resulting formulae is immediate.

It is straightforward to show that for a generic time interval Δt (not necessarily small), the function $I^{(0)}$ can be written rigorously as

$$I^{(0)}(t + \Delta t) = e^{-\beta \Delta t} I^{(0)}(t) + J^{(0)}(t, \Delta t), \quad (8)$$

where

$$J^{(0)}(t, \Delta t) \equiv \int_t^{t+\Delta t} \psi(\tau) e^{-\beta(t+\Delta t-\tau)} d\tau. \quad (9)$$

In practice, at time $t + \Delta t$, each term of the sum in equation (6) can be calculated iteratively from the values at time t , plus a contribution $J^{(0)}$ due to the star formation over the last time interval only. For example, if we adopt a standard trapezoidal rule,

$$J^{(0)}(t, \Delta t) \simeq \frac{\Delta t}{2} [\psi(t + \Delta t) + \psi(t) e^{-\beta \Delta t}], \quad (10)$$

and the final recursive formula for $R(t + \Delta t)$ is obtained by summation of the k components by using equation (8).

Remarkably, for reasons that will be explained below, the method can be extended by using in equation (5) ‘base’ functions more general than pure exponentials, namely $t^n e^{-\beta t}$, where n is an integer (and, as discussed in the appendix, not necessarily the same in all the k components). In principle, this property allows us to reproduce quite complicated time dependencies of $r_{\text{SSP}}(t)$, in particular the initial rise due to the short but finite lifetimes of the most massive stars in the IMF. For example, in the illustrative case presented in Section 3 we found it optimal to use $n = 1$ in all the components. Accordingly, we now consider

$$r_{\text{SSP}}(t) = \sum_{i=1}^k \alpha_i t e^{-\beta_i t}, \quad (11)$$

(where of course α_i and β_i are different from those appearing in equation (5), and the α_i are now in units of Gyr^{-2}). It follows that

$$R(t) = \sum_{i=1}^k \alpha_i I^{(1)}(t), \quad (12)$$

with

$$I^{(1)}(t) = \int_0^t \psi(\tau) (t - \tau) e^{-\beta(t-\tau)} d\tau. \quad (13)$$

Simple algebra shows that for a generic time interval Δt ,

$$I^{(1)}(t + \Delta t) = e^{-\beta \Delta t} [I^{(1)}(t) + I^{(0)}(t) \Delta t] + J^{(1)}(t, \Delta t), \quad (14)$$

where $I^{(0)}(t)$ is still given by equations (7) and (8) with the new α_i and β_i , and

$$J^{(1)}(t, \Delta t) \equiv \int_t^{t+\Delta t} \psi(\tau) (t + \Delta t - \tau) e^{-\beta(t+\Delta t-\tau)} d\tau. \quad (15)$$

The integral $J^{(1)}$ over the last time step can be evaluated with a simple trapezoidal rule, obtaining in this case

$$J^{(1)}(t, \Delta t) \simeq \frac{\Delta t^2}{2} \psi(t) e^{-\beta \Delta t}. \quad (16)$$

As anticipated above, the extension of the scheme of equations (8) and (14) to functions $t^n e^{-\beta t}$ with integer $n \geq 0$ is straightforward by using the binomial theorem, and the resulting recursive formula for the functions $I^{(n)}(t + \Delta t)$ involves the evaluation and the storing of the functions $I^{(j)}(t)$ for $j = 0, 1, \dots, n$. One may ask why the method works, i.e. why it is possible for this set of functions to evaluate a convolution integral over the whole SFH just by keeping track (for each of them) of $n + 1$ values relative to the previous time. The reason is easily explained following the general argument in the appendix: here we just stress that if the kernel r_{SSP} obeys a linear ordinary differential equation (ODE) with constant coefficients, then also R obeys a linear ODE with constant coefficients of the same degree, so that integration of equation (3) is equivalent to the integration of the associated ODE, and the number of needed initial data is given by the order of the equation. For example, in the case of a single function $t^n e^{-\beta t}$ the order of the ODE is $n + 1$, the number of quantities needed to be stored to evaluate $I^{(n)}$. When r_{SSP} is given by a sum of functions the argument is technically more complicated, but conceptually identical, and following the appendix one can show that the specific case of equation (11) requires the storing of $2k$ values at the previous time. For the more mathematically inclined readers, in the appendix we derive the ODE for $R(t)$ in the case of the most general combination of functions $t^n e^{-\beta t}$ with arbitrary n , and we also briefly comment on the relation of the present method with the theory of Green functions.

3 RESULTS

In the example considered here for illustrative purposes of the method, we evaluate r_{SSP} from equation (2) by using $M_{\text{ej}}(M)$ as given by Van den Hoeck & Groenewegen (1997) for low- and intermediate-mass stars ($0.8 \leq M \leq 8$), and by Woosley & Weaver (1995) for massive stars ($M > 8$). In particular, the adopted M_{ej} holds for a stellar metallicity $Z = 0.02$, very close to the concordance solar value of 0.015 (e.g. Lodders 2003). For the IMF, we adopt a Kroupa et al. (1993) $\phi(M)$ with $M_{\text{inf}} = 0.1$ and $M_{\text{sup}} = 40$, while the stellar lifetimes are taken from Padovani & Matteucci (1993). Of course, the general scheme of the method is independent of the specific choices made, and different prescriptions for the properties of the SSP can be considered as well (e.g. Ciotti et al. 1991; Pellegrini 2012).

3.1 Fitting the r_{SSP}

For the SSP described above, Fig. 1 shows the exact normalized mass return rate r_{SSP} calculated according to equation (2) (top panel,

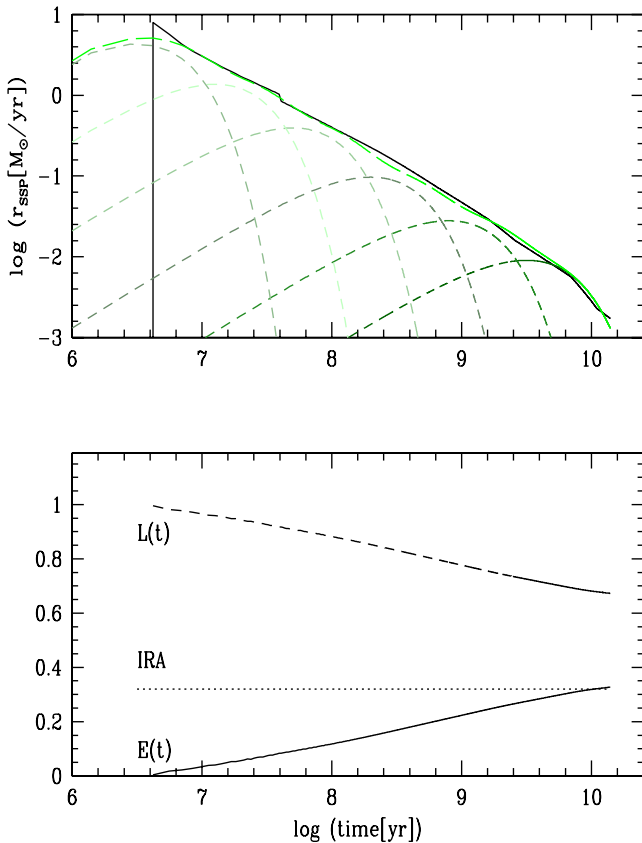


Figure 1. Top panel: the exact, normalized r_{SSP} for an SSP of solar metallicity as given by equation (2), is represented by the solid line. The long-dashed line is the analytic fit obtained from equation (11) with $k = 6$, while the six components are represented by the short-dashed lines. The fit parameters are given in Table 1. Bottom panel: the exact cumulative mass fraction $E(t)$ (solid line, equation 17), and the locked-up fraction $L(t) = 1 - E(t)$. For reference, E calculated in the IRA is represented by the dotted line.

solid line), and the corresponding returned cumulative mass fraction as a function of cosmic time (bottom panel, solid line)

$$E(t) = \int_0^t r_{\text{SSP}}(\tau) d\tau. \quad (17)$$

The bottom panel also includes the quantity $L(t) = 1 - E(t)$, i.e. the fraction of mass locked up in living stars and remnants (Portinari et al. 2004). In this example, $E(\infty) \simeq 33$ per cent and $L(\infty) \simeq 67$ per cent. For comparison, the dotted line represents the cumulative returned mass fraction calculated assuming the IRA, and it is apparent how this assumption leads to overestimate the mass return of an SSP, in particular at early times.

In Fig. 1 (top panel, short dashed lines), we also show the separate contribution of the six fit components in equation (11), whose parameters α_i and β_i are reported in Table 1, together with their sum (long dashed line). As can be seen from Table 1, the best-fitting parameters obey

$$\sum_{i=1}^6 \frac{\alpha_i}{\beta_i^2} \simeq 0.328, \quad (18)$$

i.e. the fit in equation (11) conserves the total mass almost perfectly. The plot in terms of the logarithm of time allows us to appreciate how well the fit reproduces the exact r_{SSP} , with the exception of the few Myr after the SSP formation. This occurs because the exact r_{SSP}

Table 1. Parameters of the six functions $\alpha_i t e^{-\beta_i t}$ used for the fit of the r_{SSP} considered in Section 3.1.

α_i (Gyr $^{-2}$)	β_i (Gyr $^{-1}$)
7.7624×10^{-3}	3.1623×10^{-1}
9.5499×10^{-2}	1.2589
1.3183	5.0119
21.380	19.953
2.9512×10^2	79.433
3.7153×10^3	3.1623×10^2

presents a discontinuity at the time corresponding to the lifetime of the highest mass stars (4 Myr for a $40 M_{\odot}$ star).¹

In any case, the functions $re^{-\beta t}$ allow us to reproduce the initial rise better than pure exponentials, and even better results would be obtained by using functions $t^n e^{-\beta t}$ with larger values of n . Here, for presentation purposes, we restrict to $n = 1$ as a compromise between accuracy and algebraic simplicity.

The overall impact of the initial discontinuity on the resulting mass return rate is negligible, as can be seen from Fig. 2. In the left-hand panels, we show again the exact rate r_{SSP} obtained by evaluation of equation (2) (top panel, dotted line) compared to the rate obtained by summing the six fit components (red solid line), and the relative error between the fitted function and the exact rate r_{SSP} as a function of time (bottom panel). In the right-hand panels, we show the cumulative returned mass $E(t)$ and the associated relative error for the exact r_{SSP} (dotted line), for the new method (red solid line), and in the case of IRA (blue dashed line), respectively. Note how the relative error of the new method is significantly smaller than the one obtained with the IRA, especially at early times.

3.2 Multiple stellar population

We now move to consider the case of the mass return rate of a multiple stellar population originated by a complex SFH.

In order to test the performance (in terms of computational speed and accuracy) of the new method, we consider an artificial case, characterized by a large number of star formation bursts of random amplitude. This case is relevant since it mimics the SFHs commonly encountered in SAMs, and which are typical of systems undergoing a large number of mergers and interactions, such as the ones resulting from the intricate merging trees of giant galaxies (Somerville & Kolatt 1999; Lanzoni, Mamon & Guiderdoni 2000; Calura & Menci 2009; Romano & Starkeburg 2013; Yates et al. 2013), or star formation induced by AGN activity in the inner regions of the host systems (Ciotti & Ostriker 2007). In practice, we adopt an SFH $\psi(t)$ in which, at each time step, $\log_{10} \psi$ is extracted randomly from a normal distribution with standard deviation 0.5. In the following experiment, the time step is kept constant, $\Delta t = 1.4$ Myr, suited to resolve the time contribution to the mass return rate of the most massive stars at early times (in hydrodynamical simulations considerably shorter time steps are common, down to 10^3 – 10^4 yr or less).

The behaviour of $\psi(t)$ is visualized by the blue dashed line in Fig. 3 (top-left panel), representing the mass return rate in the IRA

¹ The discontinuity in the mass return rate is due to the fact that only mass-losses from post-main-sequence phases are considered in equation (2). The discontinuity would not be present when using a function M_{ej} describing the stellar mass-losses also along the main-sequence evolution.

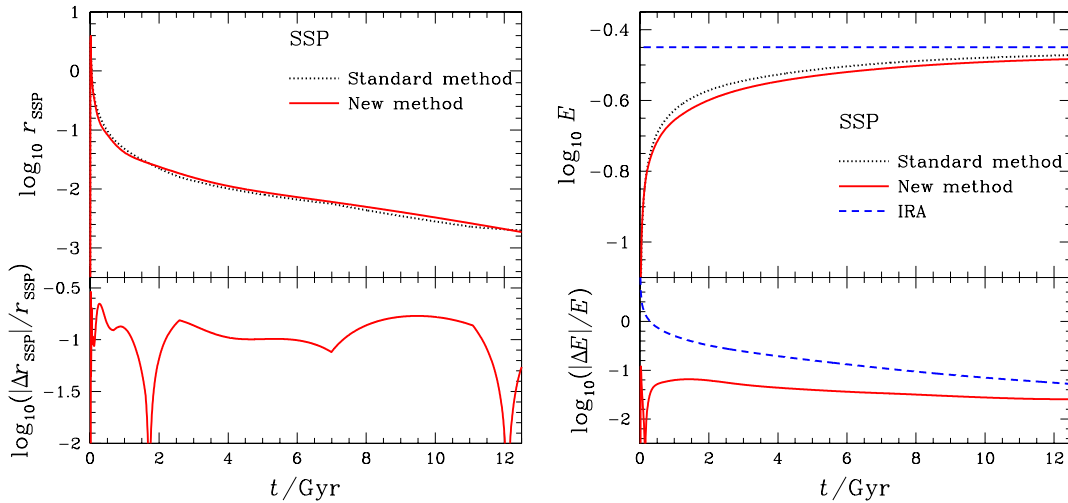


Figure 2. Left-hand panels: exact specific mass return rate r_{SSP} as given by equation (2) (dotted line), along with its fit from equation (11) for the parameters reported in Table 1 (red solid line). The relative error between the two rates is shown in the bottom panel. Right-hand panels: the cumulative ejected mass computed from the integral of the exact rate (dotted) and of the fit (solid). The dashed line represents the same quantities calculated assuming the IRA. In the bottom-right panel, the relative errors with respect to the exact rate are shown, as computed by means of the new method (solid line) and in the case of the IRA.

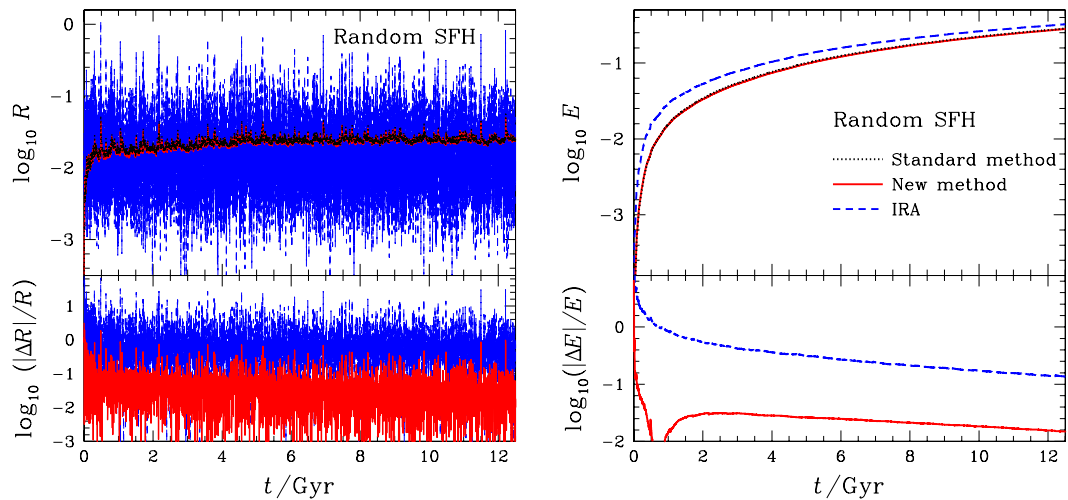


Figure 3. Left-hand panels: time evolution of the mass return rate R of a multiple stellar population originated by an SFH consisting of several bursts of random amplitude, as described in Section 3.2. The black dotted line is the rate computed with the standard method, the red line with the new method, and the blue line by using the IRA. This latter line also visualizes the adopted SFH, because from equation (4) $R_{\text{IRA}}(t) \propto \psi(t)$. The relative error of the mass return rate is shown in the bottom panel for the new method (red) and assuming the IRA (blue). Right-hand panels: the cumulative mass return E and the relative errors are shown, with the same line styles as in the left-hand panels. The quantities R and E are normalized to the total stellar mass formed over the simulation, and the time step is $\Delta t = 1.4$ Myr.

(which is proportional to ψ , see equation 4). The resulting mass return rate is also shown for the standard method and with the new method (red line): the relative errors with respect to the exact values of the mass return rate are shown in the bottom-left panel. The most striking feature is the large scatter in R_{IRA} (more than two orders of magnitude) compared to the exact rate, which instead is almost perfectly matched by the rate obtained with the new method. The relative errors represented in the bottom-left panel quantify the performance of IRA and of the new method: in general, the peaks in the relative errors of the IRA are related with peaks in the instantaneous SFH $\psi(t)$. Such peaks are also present in the mass return computed with the new method, as a consequence of the initial discontinuity following each burst, as described above.

However, the amplitude of these errors is significantly smaller than in the IRA case, as apparent.

The cumulative mass return for the multiburst SFH is shown in the right-hand panels of Fig. 3, where the quantity $E(t)$ is normalized to the stellar mass formed over the entire simulation. The top-right panel shows this quantity for the exact case, for the IRA case, and for the new method, while the bottom-right panel shows the relative error between the IRA and the exact case, and between the new method and the exact case. Note the remarkable accuracy of the new method in the multiple population case, actually even better than in the case of an SSP. This is not surprising, as for a multiple stellar population the IRA is, in practice, always in the most critical regime (just after a burst), and its error is of a

magnitude similar to that presented at early times in Fig. 2 (bottom-right panel).

3.3 Computational advantages of the new method

As already discussed, the IRA presents considerable advantages in terms of computational time and does not require any storage regarding ages and metallicities of composite stellar populations. Essentially, at any time, the mass return rate is calculated directly from the physical properties of the system at that particular time, i.e. directly from the instantaneous value of the star formation rate (see equation 4). The major problem with the IRA is that it breaks down at short times after a starburst event, when the very short characteristic heating and cooling times would lead to large overestimates of the mass return rate, which in turn can seriously affect the numerical computations by altering the available mass budget.

The method presented in this paper offers the possibility to compute the mass return rate for arbitrarily complicated SFHs, at a computational cost considerably lower than that required when the exact method is used, yet maintaining the full accuracy. The computational gain of the new method with respect to the standard method can be estimated by considering the number of needed operations. The computational time of the exact calculation scales with the total number of time steps N_{step} as $T_{\text{CPU, standard}} \propto N_{\text{step}}(N_{\text{step}} + 1)/2$, since j operations are needed to compute the mass return rate at the j th time step. With the new method, $T_{\text{CPU, new}} \propto (n + 1)kN_{\text{step}}$, because at each time the $n + 1$ integrals over the last time step $J^{(0)}, \dots, J^{(n)}$ must be computed for each of the k functions used in the fit. Therefore, $T_{\text{CPU, new}}/T_{\text{CPU, standard}} \sim 2(n + 1)k/N_{\text{step}}$: when the other parameters are fixed, the computational gain of the new method over the standard method increases linearly with N_{step} , i.e. it increases for decreasing Δt (at fixed total time span). Note that the CPU time of the new method scales with N_{step} as in the IRA.

The actual computational gain of the new method obtained in numerical experiments similar to that described in Section 3.2, where different Δt are adopted, is shown in Fig. 4, plotting the CPU time (in arbitrary units) required to evaluate the mass return for the exact method (dotted line), the new method (solid line) and the IRA (dashed line), as a function of the total number of time steps N_{step} . For our reference case with $\Delta t = 1.4$ Myr, we have $N_{\text{step}} = 8929$, $k = 6$, $n = 1$, so we expect $T_{\text{CPU, new}}/T_{\text{CPU, standard}} \simeq 2.7 \times 10^{-3}$. In fact, the experiment shows that the CPU time for the new method is a factor ≈ 500 shorter than for the standard method, in agreement with the analytic estimate. From Fig. 4 it is also apparent that the actual scaling with N_{step} nicely follows the expected scaling ($T_{\text{CPU, new}} \propto T_{\text{CPU, IRA}} \propto N_{\text{step}}$, $T_{\text{CPU, standard}} \propto N_{\text{step}}^2$). We remark again that in hydrodynamical simulations the actual Δt can easily be one or two orders of magnitude shorter than 1 Myr, so that the gain of the new method will increase accordingly; moreover, this further gain will also be multiplied by the number of grid points.

4 CONCLUSIONS

The mass returned to the ambient medium by evolving stellar populations represents a significant fraction of their initial mass (20–30 per cent for realistic IMFs), and thus being a non-negligible contribution to the evolution of stellar systems. For example, in elliptical galaxies the total mass of the returned gas is almost two orders of magnitude larger than the observed masses of central SMBHs, requiring important AGN feedback effects to prevent accretion in absence of other mechanisms able to eject the metal-rich gas from the galaxies (e.g. Ciotti & Ostriker 2001).

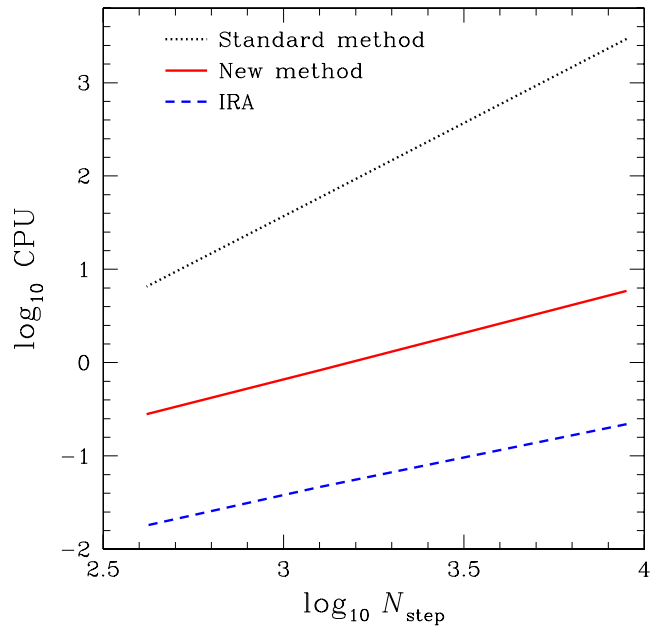


Figure 4. CPU time (in arbitrary units) required by the computation of the mass return rate by means of the standard method (dotted line), the new method (solid line) and the IRA (dashed line), as a function of the total number of time steps N_{step} . The case discussed in Section 3.2 corresponds to $\log_{10} N_{\text{step}} \simeq 3.95$.

In numerical problems dealing with the evolution of stellar populations and their mass return, when the characteristic times are longer than the stellar evolutionary times, the IRA is a useful approximation of the convolution integral describing mass return, allowing for fast and accurate computations. However, in hydrodynamical studies of star formation, of the interplay between central starbursts and AGN feedback in the co-evolution of galaxies and their central SMBHs, and of the origin of multiple stellar populations in globular clusters, the dominance of short characteristic times ($\simeq 10^3$ yr or even less), coupled with thousands (or more) spatial grid points, requires the detailed evaluation of the convolution integral. This makes the simulations excessively demanding in terms of computer memory and computational time, due to the need of storing the whole SFH at each grid point and to the number of evaluations to be performed increasing quadratically with the number of time steps.

In order to overcome these problems we developed a fast and accurate method to evaluate the mass return rate from stellar populations with arbitrarily complicated SFHs. The method presents great computational advantages over the direct evaluation of the convolution integral (the computation time-scales linearly with the number of time steps), yet its accuracy is much better than that achievable with the IRA, especially in the case of multiple populations, when the clock of the mass return rate is reset at each star formation event. The new method can be easily implemented in SAMs and in grid-based hydrodynamical simulations, and it requires to fit the mass return rate from the adopted SSP by means of simple functions, solutions of linear ODEs with constant coefficients. We have shown how these functions allow us to evaluate, at any time step, the mass return rate directly from its value at the previous time step, plus a contribution depending only on the star formation over the last time step. The general mathematical framework is presented in the appendix. In this paper, as a specific application, we have shown that the use of a linear combination of

the functions $t e^{-\beta t}$ is enough to achieve very accurate results: with arbitrarily complicated SFHs, the computed mass return rate from the resulting multiple stellar population deviates by amounts of a few per cent from the exact value, and this avoiding the storage of the SFH itself.

We conclude by noticing that the presented method, applying in general to the treatment of time convolution integrals, can be also used in problems different from that discussed in this paper, for example, to study the return of single chemical elements from multiple stellar populations (as in starbursts), or to describe the time evolution of the SNIa rate in star-forming events.

ACKNOWLEDGEMENTS

We are grateful to Laura Greggio, Francesca Matteucci, Alvio Renzini and Monica Tosi for useful comments. Financial support from PRIN MIUR 2010-2011, project ‘The Chemical and Dynamical Evolution of the Milky Way and Local Group Galaxies’, prot. 2010LY5N2T is acknowledged.

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APPENDIX A: THE GENERAL ODE FOR THE MASS RETURN RATE

For ease of notation we rewrite equation (3) as

$$Y(t) = \int_{-\infty}^t \psi(\tau) y(t - \tau) d\tau, \quad (\text{A1})$$

where the meaning of Y and y is obvious, and the lower limit of integration allows full generality in the treatment. Let assume that y is a solution of a homogeneous, constant-coefficients linear ODE of order n , i.e.

$$\sum_{i=0}^n c_i y^{(i)} = 0, \quad (\text{A2})$$

where c_i are constants, $y^{(i)} \equiv d^i y / dt^i$, and $y^{(0)} \equiv y$. Taking for granted that all the technical requirements of convergence and regularity are satisfied, mathematical induction proves that the time derivatives of the function Y can be written as

$$Y^{(i)}(t) = \sum_{j=0}^{i-1} \psi^{(i-j-1)}(t) y^{(j)}(0) + \int_{-\infty}^t \psi(\tau) y^{(i)}(t - \tau) d\tau, \quad (\text{A3})$$

for $i = 1, 2, \dots, n$. After multiplication of equation (A3) by c_i and summation over i , from equation (A2) it follows that Y also obeys the generally non-homogeneous constant-coefficients linear ODE of order n

$$\sum_{i=0}^n c_i Y^{(i)} = \sum_{i=0}^n \sum_{j=0}^{i-1} c_i \psi^{(i-j-1)}(t) y^{(j)}(0). \quad (\text{A4})$$

Note that the assumption of constant coefficients is important here, because the vanishing of the resulting linear combination of integrals at the r.h.s. of equation (A3) rests on two facts: (1) the c_i can be carried under sign of integration (this would be true also for time-dependent coefficients c_i), and (2) if $y(t)$ is a solution of equation (A2), then also $y(t - \tau)$ is a solution (this in general would not be true for time-dependent c_i). Therefore, if the kernel r_{SSP} is solution of a constant-coefficients linear ODE of order n , also the mass return rate R can be represented as a solution of constant-coefficients linear ODE of order n , and so n integration constants are required for its complete determination.

Suppose now to consider the following generalization of equations (5) and (11)

$$y(t) = \sum_{i=1}^k \sum_{j=0}^{n_i} \alpha_{ij} t^j e^{-\beta_i t}, \quad (\text{A5})$$

where $\alpha_{ij} = 0$ for $j > n_i$, $\alpha_{ij} \neq 0$ for $j = n_i$, and for $j < n_i$ the coefficients α_{ij} may be zero or not; without loss of generality, we assume that all the β_i are different. Note that in equation (A5), we consider the possibility that more than one power term t^j is associated with a given β_i , and that the maximum values n_i differ for different β_i . In this formalism, equation (5) is obtained for $n_i = 0$

and $\alpha_{i0} = \alpha_i$ ($i = 1, \dots, k$), while equation (11) for $n_i = 1$, $\alpha_{i0} = 0$ and $\alpha_{i1} = \alpha_i$ ($i = 1, \dots, k$).

We now construct the ODE for y in equation (A5), so that also the ODE for the mass return rate $R(t)$ can be explicitly obtained in this general case accordingly to equation (A4). From the theory of ODEs (e.g. Ince 1956), it is known that the function $t^n e^{-\beta t}$ is solution of a constant-coefficients linear ODE, where the root $-\beta$ of the associated characteristic polynomial has (at least) algebraic multiplicity $n + 1$. Therefore, the minimum characteristic polynomial of the ODE admitting equation (A5) as solution is

$$\prod_{i=1}^k (\lambda + \beta_i)^{n_i+1} = 0, \quad (\text{A6})$$

and the corresponding ODE of order $\sum_{i=1}^k (n_i + 1)$ is obtained by the substitution $\lambda = d/dt$.

A final comment is in order. Due to the formal similarity of equation (A1) with the convolution integral appearing in the theory of Green functions, one may ask whether $y(t)\theta(t)$ (where θ is the

Heaviside step function) is the Green function, vanishing for $t < 0$, of the ODE (A2). The answer is negative, as $y(t)$ in general lacks the needed regularity at $t = 0$ (e.g. Bender & Orszag 1978). In fact, from the request of vanishing for $t < 0$, it follows that the function $y(t)\theta(t)$ is a genuine Green function if and only if the time derivative of order $\sum_{i=1}^k (n_i + 1) - 1$ of $y(t)$ evaluated at $t = 0$ presents a finite jump, while all the remaining derivatives of $y(t)$ up to the order $\sum_{i=1}^k (n_i + 1) - 2$ evaluated at $t = 0$ vanish. These requests impose constraints on the values α_{ij} . However, in our method the α_{ij} are given as best-fitting parameters of the kernel function (in practice, by the request of reproducing r_{SSP}), so that $y(t)$ in general does not satisfy the regularity conditions at $t = 0$. However, it is easy to show that in the case of a single-component fit ($k = 1$), the function $t^n e^{-\beta t} \theta(t)$ with $n \geq 0$ is the Green function for the corresponding linear ODE of order $n + 1$.

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