CMB Temperature and Polarization

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Outline

1. CMB anisotropy calculations
2. Approximate analytic solutions
3. CMB polarization
4. Present and future cosmological constraints from the CMB

0. A (quick) review of standard cosmology

Supporting evidence

The standard cosmological model (hot big bang + inflation) is well supported by a number of observational evidences:

- the redshift of distant galaxies
- the abundances of light elements
- the existence of a cosmic microwave background (CMB)
- the age of the universe
- galaxy count evolution

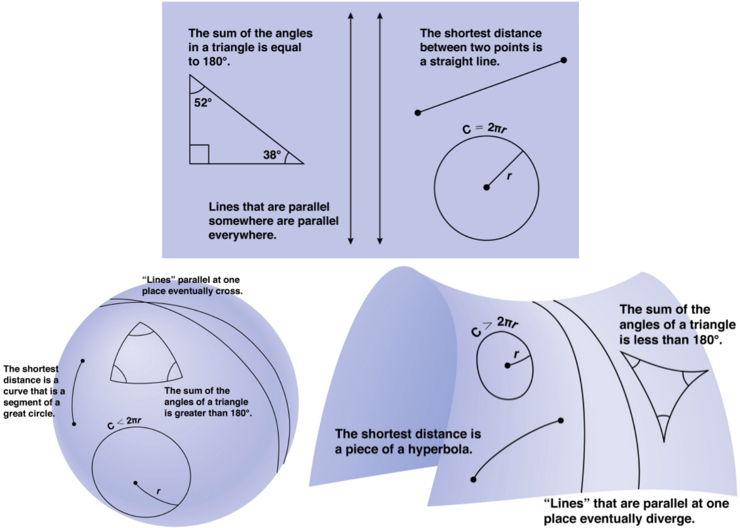
Spacetime metric

The spacetime in the standard cosmological model is described by the Robertson-Walker (RW) metric:

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \tag{1}$$

in $c = 1$ units. R has dimension of a length. The spatial coordinates r, θ and φ are comoving with free-falling observers; t is the proper time measured by such observers; k is a parameter related to the curvature of spatial sections.

Geometry



The sum of the angles in a triangle is equal to 180°.
52°
38°
Lines that are parallel somewhere are parallel everywhere.
The shortest distance between two points is a straight line.
C = 2πr
r
"Lines" parallel at one place eventually cross.
The shortest distance is a curve that is a segment of a great circle.
The sum of the angles of a triangle is greater than 180°.
C < 2πr
r
C > 2πr
r
The sum of the angles of a triangle is less than 180°.
The shortest distance is a piece of a hyperbola.
"Lines" that are parallel at one place eventually diverge.

Hubble's law

Hubble's law directly follows from the RW metric. Taking the proper distance along a radial direction ($d\theta = d\varphi = 0$) and on a geodesic ($ds = 0$) we have

$$d(t) = R(t) \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}} \tag{2}$$

so that the proper velocity between any two comoving observers is:

$$v = \dot{d} = \dot{R} \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}} = \frac{\dot{R}}{R} d \tag{3}$$

which is Hubble's law, with the definition $H \equiv \dot{R}/R$.
The present value of H is the Hubble constant H_0. We can define the Hubble time $t_H \equiv 1/H_0 = 9.78h^{-1} \times 10^{10}$ yr and the Hubble length $l_H \equiv c/H_0 = 9.25h^{-1} \times 10^{27}$ cm $= 3000h^{-1}$ Mpc.

Distances

Luminosity distance:

$$d_L^2 \equiv \frac{\mathcal{L}}{4\pi\mathcal{F}} \tag{4}$$

if a signal is emitted in $r = r_1$ at $t = t_1$ and received today $(t = t_0)$ in $r = 0$, then $\mathcal{F} = \mathcal{L}/(4\pi R^2(t_0) r_1^2 (1+z)^2)$, so that

$$d_L = R(t_0) r_1 (1+z) \tag{5}$$

Angular diameter distance:

$$d_A \equiv \frac{D}{\delta} = R(t_1) r_1 \tag{6}$$

(defined in order to preserve the Euclidean relation $\delta = D/d$).

Approximate relations

$$\frac{R(t)}{R(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \dots \tag{7}$$

with

$$H_0 \equiv \frac{\dot{R}(t_0)}{R(t_0)}, \quad q_0 \equiv -\frac{\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)} \tag{8}$$

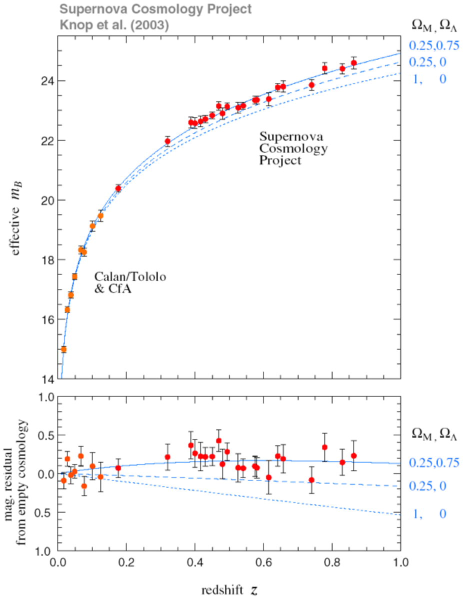
It can be shown that this leads to a non linear correction term to the Hubble law.

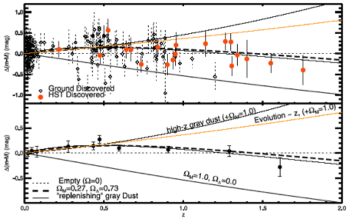
$$H_0 d_L = z + \frac{1}{2}(1 - q_0)z^2 + \dots \tag{9}$$

and similarly if we use the angular diameter distance

$$H_0 d_A = z - \frac{1}{2}(3 + q_0)z^2 + \dots \tag{10}$$

Luminosity distance from type Ia Supernovae





Cosmological stress-energy tensor

We can derive the cosmological model from Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{11}$$

$G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor and $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the curvature scalar. For an homogeneous-isotropic universe, we can assume that the stress-energy tensor $T_{\mu\nu}$ is that of an ideal fluid with energy density $\rho(t)$ and pressure $p(t)$:

$$T^{\mu\nu} = (\rho + p)\, u^{\mu} u^{\nu} - p g^{\mu\nu} \tag{12}$$

The fluid is at rest in comoving coordinates, so that $u^{\mu} = (1, 0, 0, 0)$ and

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -pg^{ij} & \\ 0 & & & \end{pmatrix} \tag{13}$$

or

$$T^{\mu}_{\ \nu} = \mathrm{diag}(\rho, -p, -p, -p) \tag{14}$$

Friedman equation

With the previous choice of $T_{\mu\nu}$ we obtain an equation for R

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{15}$$

This has to be used together with a conservation equation

$$d(\rho R^3) = -p d(R^3) \tag{16}$$

(resulting from the covariant conservation of $T_{\mu\nu}$). Combining the previous equations we get:

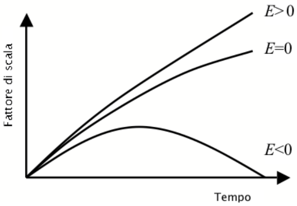
$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) \tag{17}$$

Evolution

If $k = 0, -1$, we always have $\dot{R} > 0$ and

$$\dot{R}^2 \rightarrow |k| \tag{18}$$

when $t \rightarrow \infty$ (since $d(\rho R^3) \leq 0$, and $\rho R^2 \rightarrow 0$).
When $k = +1$ we can find a value R_{max} such that $\dot{R} = 0$ for finite t, so that the universe recollapses afterwards.



Density and geometry

If $k = 0$ (flat universe) we get:

$$H^2 = \frac{8\pi G}{3}\rho \tag{19}$$

or

$$1 = \frac{8\pi G}{3H^2}\rho \equiv \frac{\rho}{\rho_c} \equiv \Omega \tag{20}$$

Friedman equation can then be rewritten as:

$$k = H^2 R^2 (\Omega - 1) \tag{21}$$

which relates the geometry of the universe and its content:

$$\begin{aligned} k = +1 &\Leftrightarrow \Omega > 1 \\ k = 0 &\Leftrightarrow \Omega = 1 \\ k = -1 &\Leftrightarrow \Omega < 1 \end{aligned} \tag{22}$$

Equation of state

Usually, density and pressure are related by

$$p = w\rho \tag{23}$$

which gives the general behaviour

$$\rho = \rho_0 \left(\frac{R}{R_0} \right)^{-3(1+w)} \tag{24}$$

and, as particular cases

$$\begin{array}{rcl} p = 0 & \Rightarrow & \rho \propto R^{-3} \;\; \text{(collisionless matter)} \\ p = \rho/3 & \Rightarrow & \rho \propto R^{-4} \;\; \text{(radiation or relativistic particles)} \end{array} \tag{25}$$

Parametric Friedman equation

We can define a density parameter for each component, so that the total density parameter is

$$\Omega = \sum_i \Omega_i. \tag{26}$$

and the Friedman equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\sum_i \Omega_{0i} a^{-3(1+w_i)} + (1-\Omega_0) a^{-2}\right] \tag{27}$$

where $a \equiv R/R_0$ and the density parameters are the present ones.

Cosmic epochs

Different components will dominate the r.h.s. at different times. For example, we can define an epoch of matter-radiation equality

$$a_{eq} \equiv \frac{\Omega_{0r}}{\Omega_{0m}} \sim 10^{-4} \qquad (28)$$

so that

$$\begin{aligned} a \gg a_{eq} \quad &\Rightarrow \quad a \propto t^{2/3} \\ a \ll a_{eq} \quad &\Rightarrow \quad a \propto t^{1/2} \end{aligned} \qquad (29)$$

In general, when a component $p = w\rho$ dominates, we have $a \propto t^{2/3(1+w)}$

Radiation density

The thermal radiation in equilibrium at temperature T has a density $\rho \propto T^4$ derived by the Planckian distribution. For an adiabatic expansion this implies $T \propto R^{-1}$.
Particles with mass m are relativistic when $kT \gg mc^2$, so we must account them in the radiation density

$$\rho = \frac{\pi^2}{30} g (kT)^4 \tag{30}$$

where

$$g = N_{\text{bosons}} + \frac{7}{8} N_{\text{fermions}} \tag{31}$$

and N is the total number of spin states. For example, for photons $g = 2$. Today, only photons and neutrinos are relativistic, so that $g \approx 3.3$.

Thermal history

From the Friedman equation during radiation domination we can derive

$$t/\mathrm{s} \approx g^{-1/2}(10^{10}\mathrm{K}/T)^2 \tag{32}$$

Using the conversion $1\mathrm{eV} \sim 1.2 \times 10^4\mathrm{K}$ we can identify different important epochs:

- today: $T \sim 2 \times 10^{-4}$ eV ($T = 2.7$ K), $g \sim 3.3$, $t \sim 13.7 \times 10^9$ years
- recombination: $T \sim 0.3$ eV, $g \sim 3.3$, $t \sim 3 \times 10^5$ years
- nucleosynthesis: $T \sim 1$ MeV, $g \sim 11$, $t \sim 1$ s
- quark-hadrons transition: $T \sim 1$ GeV, $g \sim 70$, $t \sim 2.5 \times 10^{-7}$ s
- electroweak unification: $T \sim 250$ GeV, $g \sim 100$, $t \sim 10^{-12}$ s
- grand unification (GUT): $T \sim 10^{15}$ GeV, $g \sim 100$, $t \sim 10^{-35}$ s

At earlier times, our knowledge of fundamental physics is incomplete (quantum gravity?)

Problems of the big bang model: flatness

We can recast the Friedman equation in the form

$$\Omega(t) = 1 + \frac{k}{(HR)^2} \tag{33}$$

If $\Omega = 1$ ($k = 0$) it does not change with time. When $k \neq 0$ the r.h.s. evolves. Note that:

$$\frac{d}{dt}(HR) = \ddot{R} \tag{34}$$

Thus, for "standard" components, with equation of state such that $\rho + 3p > 0$, we have $\ddot{R} < 0$ and Ω departs from unity as the universe expands. The flat solution ($\Omega = 1$) is unstable and we need extreme fine-tuning at early times in order to have $\Omega \sim 1$ today. For example:

$$\Omega(t \sim 1\mathrm{s}) = 1 \pm 10^{-17} \tag{35}$$

Problems of the big bang model: horizon

Proper horizon ($ds = 0$):

$$d_H(t) = R(t) \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}} = R(t) \int_0^t \frac{dt'}{R(t')} \tag{36}$$

It can be easily shown that, if $p = w\rho$, then

$$r_H = \frac{d_H}{R} = \frac{\alpha}{1 - \alpha}(HR)^{-1} \tag{37}$$

with $\alpha \equiv 2/(3(w + 1))$. Again, if $\rho + 3p > 0$ we have a problem, since $\alpha < 1$ and the horizon radius grows with time. This means that casual processes cannot explain the observed homogeneity of the universe at large scales.

Inflation

The previous problems are solved if:

$$\begin{aligned} \frac{d}{dt}(HR) = \frac{d}{dt}(R/l_H) &> 0 \\ \ddot{R} &> 0 \\ \rho + 3p &< 0 \end{aligned} \tag{38}$$

the three conditions are equivalent and give the kinematical definition of inflation. The prototypical inflationary behaviour is one with an exponential expansion:

$$R \propto \exp(Ht) \tag{39}$$

with H =const. It can be shown that the initial conditions problems of the big bang can be solved if $N \equiv \ln(R_f/R_i) = H(t_f - t_i) > 60$, with $t_i \sim 10^{-35}$ and $\Delta t \sim 10^{-33}$.

Cosmological constant

We can add a term to the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{40}$$

with Λ =const. The Friedman equation becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3} - \frac{k}{R^2} + \frac{8\pi G}{3}\rho \tag{41}$$

If Λ dominates the r.h.s., we have $H^2 = \Lambda/3$ =const. and $R \propto \exp\left(\sqrt{\Lambda/3}t\right)$ (de Sitter solution).

Vacuum energy

The cosmological constant can also be moved to the r.h.s. of Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{42}$$

By comparing the new term with the ideal fluid stress-energy tensor, we see that this is equivalent to the introduction of a new component with

$$\begin{aligned} p &= -\Lambda/8\pi G \\ \rho &= \Lambda/8\pi G \end{aligned} \tag{43}$$

or equation of state $p = -\rho$. Since in this case the stress-energy tensor does not depend on u_μ it has the interpretation of the stress-energy tensor of empty space (vacuum).

Vacuum energy and inflation

We can then realize inflation if the vacuum has the right energy for a suitable time interval, for example making a (slow) transition from a false vacuum to a true vacuum.
Cosmological observations constrain the present vacuum energy to be

$$\Omega_\Lambda \sim 0.7 \Rightarrow \rho_\Lambda \sim 10^{-44}\mathrm{GeV}^4 \tag{44}$$

much smaller than any natural energy scale from theoretical physics:

$$\begin{array}{rll} \mathrm{GUT}: & T_{GUT} \sim 10^{15}\mathrm{GeV} & \Rightarrow \rho_V \sim T_{GUT}^4 \sim 10^{60}\mathrm{GeV}^4 \\ \mathrm{EW}: & T_{EW} \sim 10^{2}\mathrm{GeV} & \Rightarrow \rho_V \sim T_{EW}^4 \sim 10^{8}\mathrm{GeV}^4 \end{array} \tag{45}$$

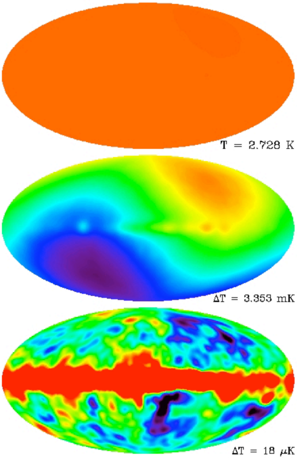
so it should either be exactly zero today, after inflation, or some mechanism should explain its extremely small value (fine-tuning).

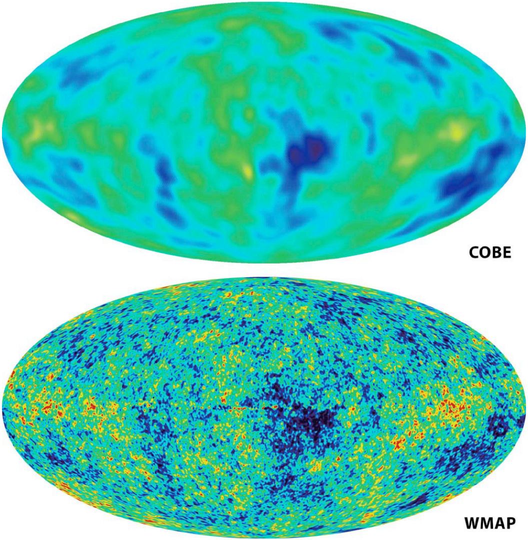
1. CMB anisotropy calculations

Some history

- Lemaître, circa 1927: radiation must fill the universe
- Gamow, Alpher, Hermann, circa 1950: blackbody background T $\sim$ 10K
- Herzberg 1950: T= 2.3 K from interstellar cyanogen
- Ohm 1961, Doroshkevich & Novikov 1963
- Penzias & Wilson 1964, Dicke, Peebles, Roll & Wilkinson 1965

CMB anisotropy





CMB statistics

Spherical harmonic expansion:

$$\frac{\Delta T}{T}(\vec{x}_0, \theta, \phi) = \sum_{lm} a_{lm}(\vec{x}_0) Y_{lm}(\theta, \phi)$$

Angular power spectrum:

$$C_l \equiv \langle |a_{lm}|^2 \rangle, \qquad \tilde{C}_l(\vec{x}_0) = \sum_{m=-l}^{l} \frac{|a_{lm}(\vec{x}_0)|^2}{2l+1}$$

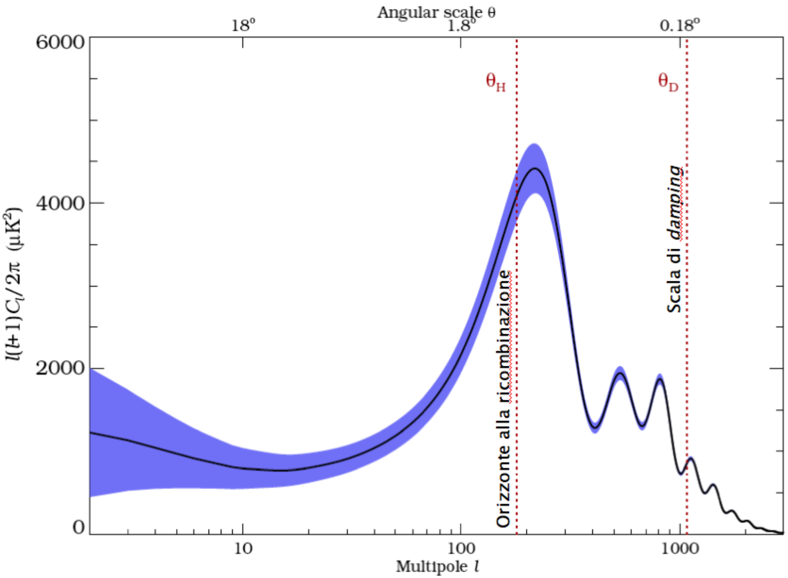
Two-point correlation function:

$$C(\alpha) \equiv \langle \frac{\Delta T}{T}(\hat{\gamma}) \frac{\Delta T}{T}(\hat{\gamma} + \alpha) \rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\alpha)$$

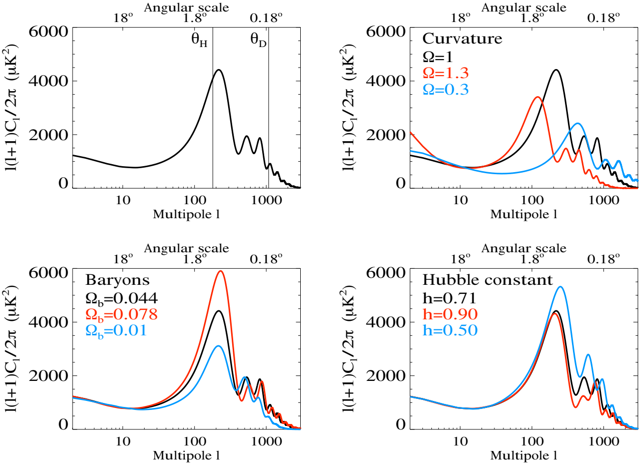
Primordial power spectrum:

$$\langle \delta^\star(\vec{k}) \delta(\vec{k}') \rangle \equiv (2\pi)^3 \delta_D(\vec{k} - \vec{k}') P(k), \qquad P(k) = A k^n$$

CMB angular power spectrum



Parameter dependence



Primordial plasma

Photons + free electrons in thermodynamical equilibrium.
Thomson scattering:

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} |\hat{\epsilon}' \cdot \hat{\epsilon}|^2 \qquad (\sigma_T = 8\pi\alpha^2/3m_e)$$

Optical depth:

$$\tau(\eta) = -\int_{\eta_0}^{\eta} d\eta' \sigma_T n_e(\eta') ca \qquad (d\eta \equiv dt/a)$$

Mean free path:

$$\dot{\tau}^{-1} = (\sigma_T n_e a)^{-1}$$

Visibility function:

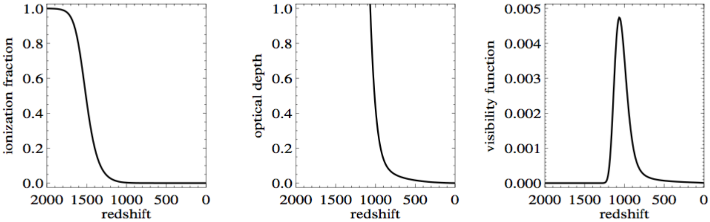
$$f(\eta) = -\dot{\tau} e^{-\tau}$$

Recombination

The fraction of free electrons, $X_e \equiv n_e/n_H$, can be computed using Saha equilibrium equation:

$$\frac{X_e^2}{1 - X_e} = \frac{(2m_e kT)^{3/2}}{n_H} e^{-B/kT} \qquad (B = 13.6\,\text{eV})$$

Hydrogen recombination occurs relatively rapidly at $a_\star \approx 10^{-3}$, $kT \approx 1/3$ eV.



Perturbation theory

Linear perturbations to a flat background.
Synchronous gauge:

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

Conformal Newtonian gauge:

$$ds^2 = a^2(\eta)\left[-(1+2\psi)d\eta^2 + (1-2\phi)dx^i dx_i\right]$$

Linearized Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}$$

$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \Leftarrow$ metric (gravity)
$T_{\mu\nu} \Leftarrow$ components (baryons, photons, neutrinos, dark matter, dark energy)

Relativistic components

Photon distribution function:

$$f(\vec{x}, \vec{p}, t) = \frac{\delta N}{d^3x\, d^3p}$$

Liouville theorem $\Rightarrow$ f is conserved along the particle trajectory:

$$\frac{df}{dt} = \frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial t} + \frac{\partial f}{\partial p^\mu}\frac{\partial p^\mu}{\partial t} = 0$$

But: collisions are important before recombination:

$$\frac{df}{dt} = C[f]$$

Boltzmann equation

Peebles & Yu (1970):

$$\frac{\partial \iota}{\partial t} + \frac{\gamma_\alpha}{a}\frac{\partial \iota}{\partial x^\alpha} - 2\gamma_\alpha\gamma_\beta + \frac{\partial h_{\alpha\beta}}{\partial t} = \sigma_T n_e \left(\delta_\gamma + 4\gamma_\alpha v^\alpha - \iota\right)$$

where:

$$\int dp\, p^3 f \equiv I = \bar{\rho}_\gamma \left[1 + \iota(\theta, \phi)\right]/4\pi$$

$$\delta_\gamma = \int d\Omega\, \iota/4\pi$$

$$\gamma_1 = \sin\theta\cos\phi, \quad \gamma_2 = \sin\theta\sin\phi, \quad \gamma_3 = \cos\theta$$

Plane wave expansion

$$\iota = \iota(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \iota(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}}$$

Choose $\hat{x}_3$ aligned with $\vec{k}$:

$$\gamma_\alpha k^\alpha = k \cos\theta \equiv k\mu$$

$$h_{11} = h_{22} \Rightarrow h = h_{11} + h_{22} + h_{33} = 2h_{11} + h_{33}$$

Then:

$$\frac{\partial \iota}{\partial t} + i\frac{kc}{a}\mu\iota + \left(1 - 3\mu^2\right)\frac{\partial h_{33}}{\partial t} - \left(1 - \mu^2\right)\frac{\partial h}{\partial t} = \sigma_T n_e c \left(\delta_\gamma + 4\mu\frac{v}{c} - \iota\right)$$

Some definitions

conformal time	$d\eta \equiv dt/a, \quad \dot{f} \equiv \partial f/\partial\eta$
temperature fluctuation	$\Delta(\mu,k,\eta) \equiv \iota(\mu,k,\eta)/4$
	$\Delta(\mu,k,\eta) \equiv \sum_l (-i)^l (2l+1)\Delta_l(k,\eta)P_l(\mu)$
energy density	$\delta_\gamma \equiv \int_{-1}^{1} \frac{d\mu}{2}\iota = 4\Delta_0$
velocity	$\beta \equiv v/c$
gravity	$\mathcal{H}(\mu,k,\eta) \equiv \left(1-3\mu^2\right)\dot{h}_{33} - \left(1-\mu^2\right)\dot{h}$
baryon/photon ratio	$\mathcal{R} \equiv \frac{3}{4}\frac{\rho_b}{\rho_\gamma}$

Computing the anisotropy

The evolution of temperature fluctuations is governed by a set of linear equations

$$\dot{\Delta} + ikc\mu\Delta + \mathcal{H} = -\dot{\tau}\left(\Delta_0 + \mu\beta - \Delta\right)$$

$$\dot{\beta} + \frac{\dot{a}}{a}\beta = \frac{\dot{\tau}}{\mathcal{R}}\left(3i\Delta_1 + \beta\right)$$

$$\dot{\delta}_b = -kc\beta + \frac{\dot{h}}{2}$$

$$\dot{\delta}_b = \frac{\dot{h}}{2}$$

+ gravitational potential equations.
Initial conditions, e.g. adiabatic (iso-enthropric):

$$S = \frac{4}{3}\frac{a}{n_b}T^3 \Rightarrow \frac{\delta S}{S} = \frac{3}{4}\delta_\gamma - \delta_b \Rightarrow \delta_b = \frac{3}{4}\delta_\gamma$$

Numerical integration

$$\dot{\Delta} + ikc\mu\Delta + \mathcal{H} = -\dot{\tau}\left(\Delta_0 + \mu\beta - \Delta\right)$$

One approach is to solve for Δ_l, arbitrary l (old Boltzmann codes): very slow!
Fast, accurate codes (CMBFAST, CAMB) are based on the line of sight solution (Seljak & Zaldarriaga, 1996):

$$\frac{d}{d\eta}\left[\Delta e^{-\tau} e^{ikc\mu\eta}\right] = \left[-\dot{\tau}\left(\Delta_0 + \mu\beta\right) - \mathcal{H}\right] e^{-\tau} e^{ikc\mu\eta}$$

$$\Rightarrow \Delta(\eta_0) e^{ikc\mu\eta_0} = -\int_0^{\eta_0} d\eta \left[f(\eta)\left(\Delta_0 + \mu\beta\right) e^{ikc\mu\eta}\right] - \mathcal{H} e^{-\tau} e^{ikc\mu\eta}$$

$$C_l = \frac{2}{\pi}\int dk\, k^2 P(k) |\Delta_l|^2$$

2. Approximate analytic solutions

Instantaneous recombination

Sharply peaked visibility function:

$$f(\eta) = \delta(\eta - \eta_\star), \quad e^{-\tau} = \begin{cases} 1, & \eta \geq \eta_\star \\ 0, & \eta < \eta_\star \end{cases}$$

$$\Rightarrow \Delta(\eta_0) = \left[\Delta_0(\eta_\star) + \mu\beta(\eta_\star)\right] e^{ikc\mu(\eta_\star - \eta_0)} - \int_{\eta_\star}^{\eta_0} d\eta\, \mathcal{H} e^{ikc\mu(\eta - \eta_0)}$$

Use:

$$e^{ikc\mu(\eta_\star - \eta_0)} = \sum_l (-i)^l (2l+1) j_l \left[kc(\eta_\star - \eta_0)\right] P_l(\mu)$$

$$\mu e^{ikc\mu x} = \sum_l (-i)^l (2l+1) j_l'(kcx) P_l(\mu)$$

Solution

$$\Rightarrow \Delta_l(\eta_0) = \Delta_l^0(\eta_0) + \Delta_l^1(\eta_0) + \Delta_l^{\rm ISW}(\eta_0)$$

where:

$$\begin{aligned}
\Delta_l^0(\eta_0) &= \Delta_0(\eta_\star) j_l\left(kD_\star\right) \\
\Delta_l^1(\eta_0) &= \beta(\eta_\star) j_l'\left(kD_\star\right) \\
&= \beta(\eta_\star)\left[j_{l-1}(kD_\star) - l(kD_\star)^{-1} j_l(kD_\star)\right] \\
\Delta_l^{\rm ISW}(\eta_0) &= -\int_{\eta_\star}^{\eta_0} d\eta \int_{-1}^{1} \frac{d\mu}{2} \mathcal{H} j_m\left[kc(\eta-\eta_0)\right] P_l(\mu) P_m(\mu)
\end{aligned}$$

and $D_\star = c(\eta_\star - \eta_0)$ is the distance to last scattering surface.

Tight coupling limit

At early times, the photon mean free path is negligible ($\dot{\tau}^{-1} \ll 1$). Consider:

$$\Delta_0 + \mu\beta - \Delta = \dot{\tau}^{-1}\left(\dot{\Delta} + ikc\mu\Delta + \mathcal{H}\right)$$
$$3i\Delta_1 + \beta = \mathcal{R}\dot{\tau}^{-1}\left(\dot{\beta} + \frac{\dot{a}}{a}\beta\right) \quad (46)$$

Zero-th order in $\dot{\tau}^{-1}$:

$$\beta = -3i\Delta_1 \quad (47)$$
$$\Delta = \Delta_0 + \mu\beta = \Delta_0 - 3i\mu\Delta_1$$

First order:

$$\Delta = \Delta_0 + \mu\beta + \dot{\tau}^{-1}\left[\dot{\Delta}_0 + \mu\dot{\beta} + ikc\mu\Delta_0 + ikc\mu^2\beta + \mathcal{H}\right]$$

Tight coupling limit

Integrate last equation over $\int d\mu/2$:

$$\Rightarrow \dot{\Delta}_0 + kc\Delta_1 + \int \frac{d\mu}{2}\mathcal{H} = 0 \tag{48}$$

and over $\int \mu\, d\mu/2$:

$$-i\Delta_1 = \frac{\beta}{3} + \dot{\tau}^{-1}\left[\frac{\dot{\beta}}{3} + i\frac{kc}{3}\Delta_0\right]$$

Eq. (46) gives:

$$3i\Delta_1 + \beta = \mathcal{R}\dot{\tau}^{-1}\left[\dot{\beta} + \frac{\dot{a}}{a}\beta\right]$$

Combining, and using Eq. (47), we get:

$$(1+\mathcal{R})\dot{\Delta}_1 + \frac{\dot{a}}{a}\mathcal{R}\Delta_1 - \frac{kc}{3}\Delta_0 = 0$$

Tight coupling limit

Finally, using Eq. (48) and introducing the sound speed $c_s^2 \equiv c^2/[3(1+\mathcal{R})]$, we get:

$$\ddot{\Delta}_0 + \frac{\dot{a}}{a}\frac{\mathcal{R}}{1+\mathcal{R}}\dot{\Delta}_0 + k^2 c_s^2 \Delta_0 = -\int \frac{d\mu}{2}\left[\dot{\mathcal{H}} + \frac{\dot{a}}{a}\frac{\mathcal{R}}{1+\mathcal{R}}\mathcal{H}\right]$$

Note that $\int d\mu/2\,\mathcal{H} = \dot{h}/6$. Defining:

$$\Phi \equiv \frac{1}{2k^2}\left[\dot{h} + \frac{\dot{a}}{a}\frac{\mathcal{R}}{1+\mathcal{R}}h\right]$$

we can write:

$$\ddot{\Delta}_0 + \frac{\dot{a}}{a}\frac{\mathcal{R}}{1+\mathcal{R}}\dot{\Delta}_0 + k^2 c_s^2 \Delta_0 = -\frac{k^2}{3}\Phi$$

Acoustic oscillations

Limit $\dot{a}/a \to 0$, $\Phi =$ constant:

$$\ddot{\Delta}_0 + k^2 c_s^2 \Delta_0 = -\frac{k^2}{3}\Phi$$

Note that in this limit Φ is the Newtonian gravitational potential.
Forced harmonic oscillator:

$$(1+\mathcal{R})\ddot{\Delta}_0 + \frac{k^2 c^2}{3}\Delta_0 = -\frac{k^2}{3}(1+\mathcal{R})\Phi$$

With the initial condition $\Delta_0(0) = -2\Phi/3c^2$ we get the solution:

$$\Delta_0(\eta) = (1/3+\mathcal{R})\frac{\Phi}{c^2}\cos(kc_s\eta) - (1+\mathcal{R})\frac{\Phi}{c^2}$$

Damping

Tight coupling breaks down near recombination (Silk, 1968).
Comoving mean free path:

$$L = \dot{\tau}^{-1} = (\sigma_T n_e a)^{-1}$$

Number of collisions in $d\eta$:

$$c d\eta / L = \sigma_T n_e a \, d\eta = \dot{\tau} d\eta$$

Distance traveled by random walk (damping scale):

$$d\lambda_D^2 = (\dot{\tau} d\eta) \, L^2 = c^2 \dot{\tau}^{-1} d\eta$$

Perturbations are damped by a factor e^{-k^2/k_D^2}, where:

$$k_D^{-2} \sim \lambda_D^2 \sim c^2 \int d\eta \, \dot{\tau}^{-1}$$

Angular power spectrum

$$C_l = \frac{2}{\pi} \int dk\, k^2 P(k) |\Delta_l|^2$$

Consider the monopole contribution:

$$\Delta_l^0(\eta_0) = \Delta_0(\eta_\star) j_l \left[k D_\star \right]$$

$j_l(x)$ is sharply peaked at $l \sim x$, so that the main contribution to the power spectrum comes from:

$$k D_\star \approx l$$

This sets a one-to-one relation between real and angular space.

Acoustic peaks

Acoustic oscillations:

$$\Delta_0(\eta) = (1/3 + \mathcal{R})\frac{\Phi}{c^2}\cos(kc_s\eta) - (1 + \mathcal{R})\frac{\Phi}{c^2}$$

Contribution to the power spectrum:

$$|\Delta_l(\eta_0)|^2 \sim |\Delta_0(\eta_\star)|^2 j_l^2(kD_\star)$$

Peaks at:

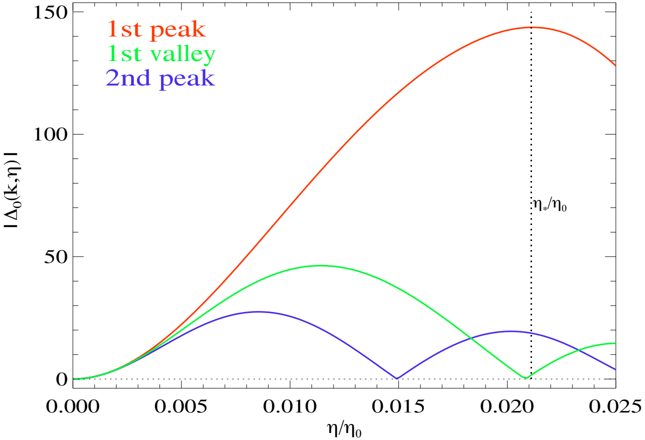
$$k_m = mk_A, \quad k_A \equiv \frac{\pi}{c_s\eta_\star} \equiv \frac{\pi}{S_\star}, \quad m = 1, 2, \ldots$$

or:

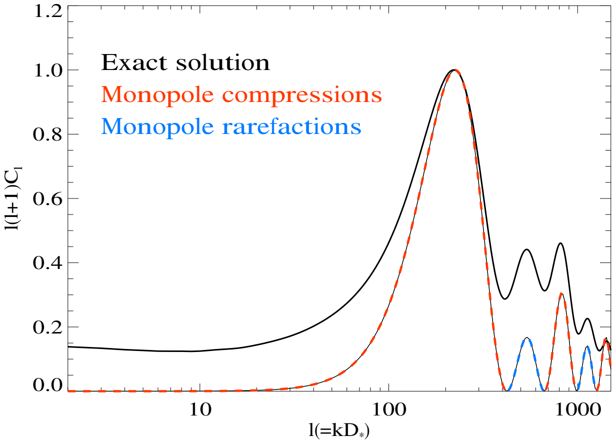
$$l_m = ml_A, \quad l_A = k_A D_\star = \pi\frac{D_\star}{S_\star}$$

where $S_\star$ is the sound horizon at decoupling.

Acoustic peaks



Acoustic peaks



Geometry

In a curved universe, the distance to the last scattering surface is different from the flat case, because of geodesic deviation:

$$D'_\star = R_0 S(D_\star / R_0)$$

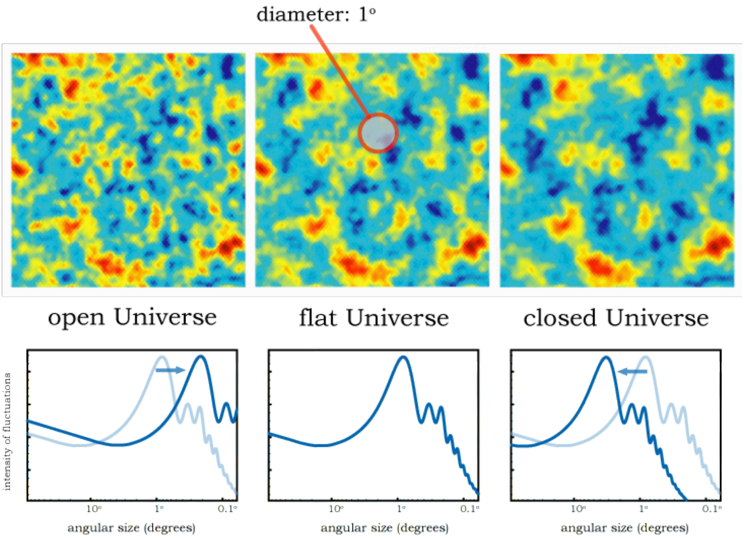
where $R_0 = cH_0^{-1}|\Omega - 1|^{-1/2}$ and:

$$S(r) \equiv \begin{cases} \sin r, & \Omega > 1 \\ r, & \Omega = 1 \\ \sinh r, & \Omega < 1 \end{cases}$$

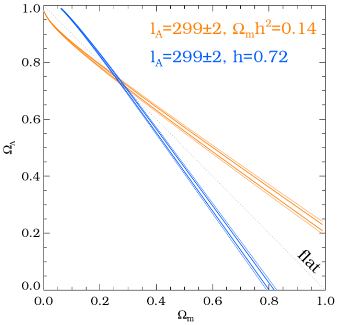
$D_\star$ itself also changes:

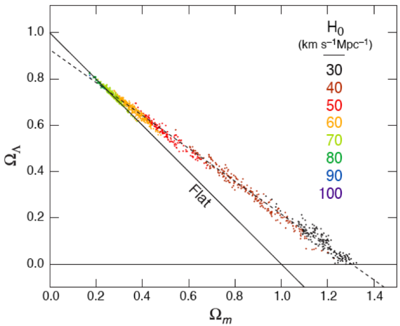
$$D_\star = cH_0^{-1} \int_1^{a_\star} \frac{da}{\left[\Omega_r + \Omega_m a + (1-\Omega)a^2 + \Omega_{de} a^{(1-3w)}\right]^{1/2}}$$

Geometry


diameter: 1°
open Universe
flat Universe
closed Universe
intensity of fluctuations
10°
1°
0.1°
angular size (degrees)
angular size (degrees)
angular size (degrees)

Geometrical degeneracy





Summary of parameter dependence

- Acoustic scale l_A controls peaks position: depends on Ω, Λ, w, Ω_m, h.
- Enhancing baryon density $\Omega_b h^2$ (through $\mathcal{R}$) enhances ratio of odd/even peaks height.
- Enhancing matter content, $\Omega_m h^2$, reduces peaks height
- Enhancing the cosmological constant Λ, enhances low l power through the integrate Sachs-Wolfe effect (more on this later)
- Spectral index of primordial scalar perturbations, n, tilts the spectrum
- Reionization, i.e. $\tau \neq 0$ today, damps high l power by roughly $e^{-2\tau}$

3. CMB polarization

Polarization basics

Nearly monochromatic plane wave:

$$E_x = a_x(t)\cos(\omega_0 t + \theta_x(t)), \qquad E_y = a_y(t)\cos(\omega_0 t + \theta_y(t))$$

Stokes parameters:

$$\begin{aligned} I &= \langle a_x\rangle^2 + \langle a_y\rangle^2 \\ Q &= \langle a_x\rangle^2 - \langle a_y\rangle^2 \\ U &= \langle 2a_x a_y \cos(\theta_x - \theta_y)\rangle \\ V &= \langle 2a_x a_y \sin(\theta_x - \theta_y)\rangle \end{aligned}$$

Averages $\langle\cdot\rangle$ are over long times (compared to the inverse frequency of the wave).

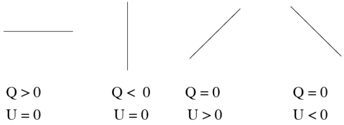
Physical meaning

Complete linear polarization:

- along $\hat{x}$ $(a_y = 0) \Rightarrow Q = I$, $U = V = 0$
- along $\hat{y}$ $(a_x = 0) \Rightarrow Q = -I$, $U = V = 0$
- at $\pi/4$ with respect to $\hat{y}$ $(a_x = a_y,\ \theta_x = \theta_y) \Rightarrow U = I$, $Q = V = 0$
- at $-\pi/4$ with respect to $\hat{y}$ $(a_x = a_y,\ \theta_x = \pi + \theta_y) \Rightarrow U = -I$, $Q = V = 0$

Complete circular polarization:

- Anti-clockwise $(a_x = a_y,\ \theta_x = \pi/2 + \theta_y) \Rightarrow Q = U = 0$, $V = I$
- Clockwise $(a_x = a_y,\ \theta_x = -\pi/2 + \theta_y) \Rightarrow Q = U = 0$, $V = -I$



Transformation laws

Under rotation of the $x - y$ plane trough an angle ϕ:

$$\begin{aligned} Q' &= Q\cos 2\phi + U\sin 2\phi \\ U' &= -Q\sin 2\phi + U\cos 2\phi \end{aligned}$$

while I and V are invariant.
A polarization "vector" $\vec{P}$ can be defined, with invariant magnitude $P \equiv Q^2 + U^2$ and orientation $\alpha \equiv 1/2\tan^{-1} U/Q$ (transforming as $\alpha' = \alpha - \phi$).

Polarization from Thomson scattering

Unpolarized incident nearly monochromatic plane wave (with $I'_x \equiv \langle a'_x \rangle^2$, $I'_y \equiv \langle a'_y \rangle^2$ and $I'_x = I'_y = I'/2$), with cross sectional area σ_B, scattered along direction $\hat{z}$:

$$\begin{aligned}
\frac{dI_x}{d\Omega} &= \frac{3\sigma_T}{8\pi\sigma_B}\left(I'_x|\hat{\epsilon_x}' \cdot \hat{\epsilon}_x|^2 + I'_y|\hat{\epsilon_y}' \cdot \hat{\epsilon}_x|^2\right) = \frac{3\sigma_T}{16\pi\sigma_B} I' \\
\frac{dI_y}{d\Omega} &= \frac{3\sigma_T}{8\pi\sigma_B}\left(I'_x|\hat{\epsilon_x}' \cdot \hat{\epsilon}_y|^2 + I'_y|\hat{\epsilon_y}' \cdot \hat{\epsilon}_y|^2\right) = \frac{3\sigma_T}{16\pi\sigma_B} I' \cos^2\theta
\end{aligned}$$

where we used $\hat{\epsilon}'_x \cdot \hat{\epsilon}_x = 1$, $\hat{\epsilon}'_y \cdot \hat{\epsilon}_x = 0$, $\hat{\epsilon}'_y \cdot \hat{\epsilon}_y = \cos\theta$ (from choice of reference frame). Then:

$$\begin{aligned}
dI &= dI_x + dI_y = \frac{3\sigma_T}{16\pi\sigma_B} I'(1 + \cos^2\theta) d\Omega \\
dQ &= dI_x - dI_y = \frac{3\sigma_T}{16\pi\sigma_B} I' \sin^2\theta d\Omega
\end{aligned}$$

By simmetry, no V polarization can be generated.

Polarization from Thomson scattering

Integrate over all incoming directions:

$$I(\hat{z}) = \frac{3\sigma_T}{16\pi\sigma_B}\int d\Omega\,(1+\cos^2\theta)I'(\theta,\phi)$$
$$Q(\hat{z}) - iU(\hat{z}) = \frac{3\sigma_T}{16\pi\sigma_B}\int d\Omega\,\sin^2\theta e^{2i\phi}I'(\theta,\phi)$$

Expand incoming radiation, $I' = \sum a_{lm}Y_{lm}$:

$$I(\hat{z}) = \frac{3\sigma_T}{16\pi\sigma_B}\left[\frac{8}{3}\sqrt{\pi}a_{00} + \frac{4}{3}\sqrt{\frac{\pi}{5}}a_{20}\right]$$
$$Q(\hat{z}) - iU(\hat{z}) = \frac{3\sigma_T}{4\pi\sigma_B}\sqrt{\frac{2\pi}{15}}a_{22}$$

Thus, polarization is generated if incident radiation has a quadrupole anisotropy.

Polarization from CMB anisotropy

In the tight coupling limit, no quadrupole anisotropy can be generated.
Near recombination tight coupling breaks down.
Quadrupole equation:

$$\dot{\Delta}_2 = kc\frac{2}{3}\Delta_1 - \dot{\tau}\Delta_2$$

First order in $\dot{\tau}^{-1}$:

$$\Delta_2 = \dot{\tau}^{-1}kc\frac{2}{3}\Delta_1 = \dot{\tau}^{-1}kc\frac{2i}{9}\beta$$

Polarization from CMB anisotropy

In terms of the damping scale $k_D^{-2} \sim c^2 \eta_\star \dot{\tau}^{-1}$:

$$\Delta_2 \sim \left(\frac{k}{k_D}\right) \frac{\beta}{k_D c \eta_\star} \sim \left(\frac{l}{l_D}\right) \frac{\beta}{k_D c \eta_\star}$$

So, polarization peaks at $l \sim l_D$ and vanishes for $l > l_D$ because of the suppression of the source due to damping.
Furthermore, polarization is in phase with velocity. Remember that $\dot{\Delta}_0 = -kc\Delta_1 = ikc\beta/3$ (neglecting the gravitational term), so that $\beta \sim \sin(kc_s\eta)$. This implies that polarization peaks are $\pi/2$ out of phase with temperature.
Finally, polarization is generated just near recombination.

Polarization statistics

The quantity $Q \pm iU$ transforms as a spin 2 object:

$$Q \pm iU \rightarrow (Q \pm iU)e^{\mp 2i\phi}$$

Then, it can be expanded using spin spherical harmonics ${}^{s}Y_{lm}$:

$$a^E_{lm} \pm i a^B_{lm} = -\int d\hat{n}\ {}^{\pm 2}Y^*_{lm}[Q(\hat{n}) \pm iU(\hat{n})]$$

thus identifying the scalar and pseudo-scalar fields $E(\hat{n}) \equiv \sum a^E_{lm} Y(\hat{n})$ and $B(\hat{n}) \equiv \sum a^B_{lm} Y(\hat{n})$.
The generalization of the power spectrum is:

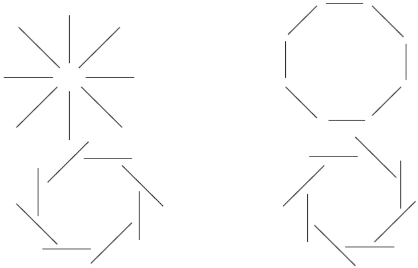
$$\langle a^{X,*}_{lm} a^{X'}_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C^{XX'}_l$$

where $X, X' \in \{T, E, B\}$.

E and B modes

Pure E

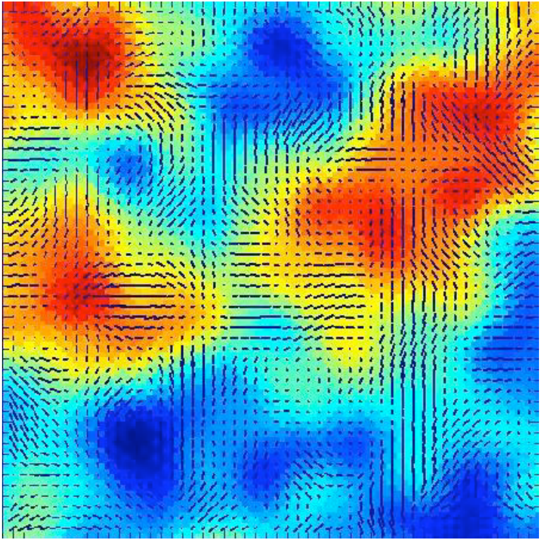
Pure B



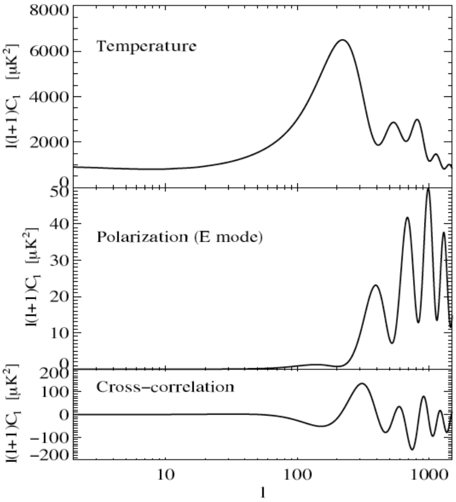
CMB polarization features

- Test of acoustic oscillations
- Test of recombination physics
- Signature from gravity waves
- Signature from reionization
- Breaking of parameter degeneracies

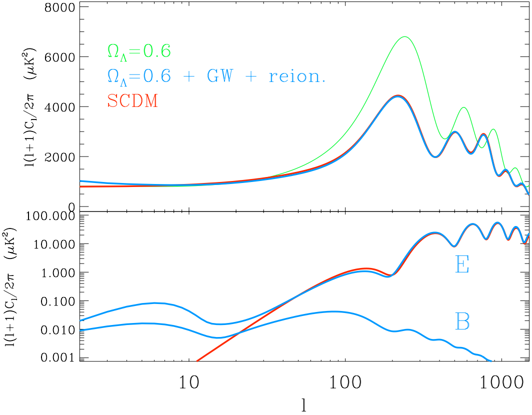
Polarization pattern



Polarization power spectra

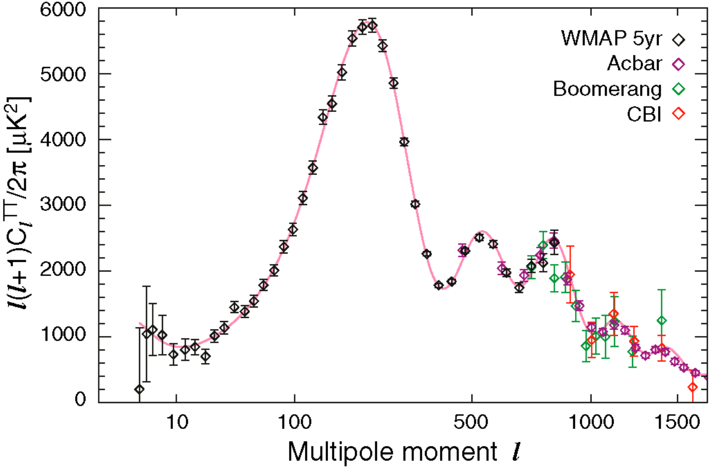


Breaking the degeneracies

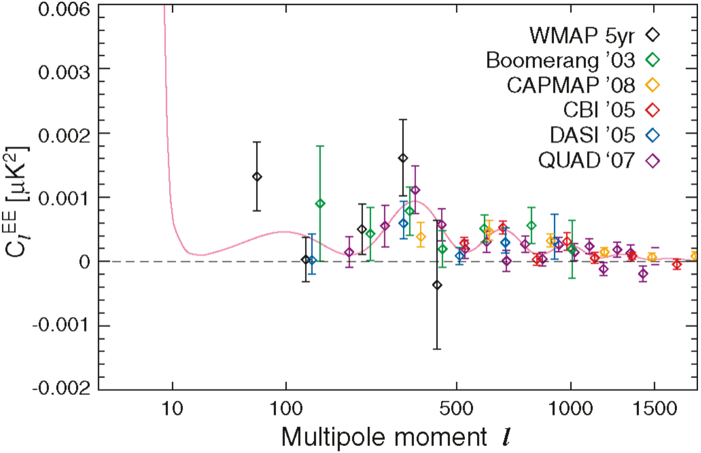


4. Present and future cosmological constraints from the CMB

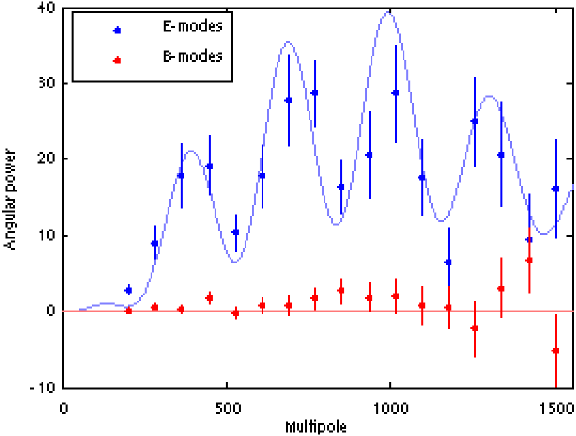
WMAP power spectrum



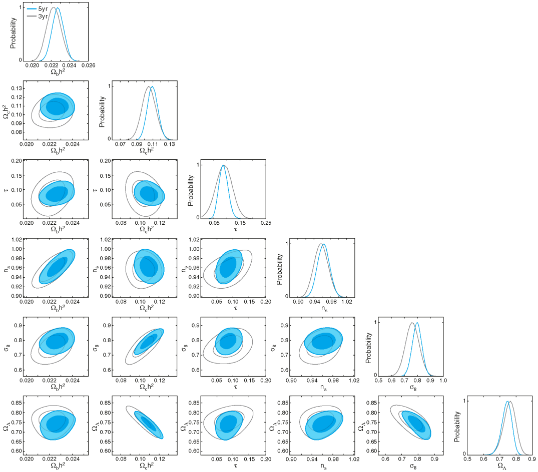
WMAP TE power spectrum



QUAD E and B power spectrum



WMAP parameters



(from Dunkley et al 2008)