Chapter 8

Cosmological perturbations

Understanding the formation of the large-scale structure of the Universe has become one of the most important tasks for modern cosmology. Many fundamental questions, ranging from the physical nature of dark matter to the formation of temperature anisotropies of the CMB and the determination of the epoch of galaxy formation, join together here.

We presently believe that large-scale structures formed due to gravitational instability of a stochastic distribution of primordial density fluctuations possibly generated during an early inflationary epoch. The Newtonian limit of general relativity is, in general, sufficiently accurate to describe the evolution of density perturbations at a basic level. However, even the simplest discussion requires some knowledge of statistics.

In this chapter we introduce some basic concepts of the theory of random fields. Historically, the statistical analysis of random fields was first developed to discuss electrical noise in communication devices. Nowadays it represents a well mature branch of mathematical statistics which finds a broad spectrum of applications in many problems of physics, engineering and economics.

8.1 Random fields

8.1.1 Random variables

In nature there exist some systems for which it is not possible to predict the evolution (for instance because of the incomplete knowledge of the initial conditions and (or) some other unknown factors). Typical examples are flipping a coin or rolling a dice. From the mathematical point of view, these systems are described through random variables. A random (or stochastic) variable, $\mathcal{S}$, is determined by a set of numbers (its possible values) and a probability measure defined over this set. The idea is to consider a large (formally infinite) collection of identical systems (called, respectively, the *ensemble* and its *realizations*) and build the cumulative probability distribution function $I_{\mathcal{S}}(\alpha)$ by computing the fraction of systems in the ensemble

for which $\mathcal{S}$ assumes values smaller than α. Denoting the operation of averaging over the ensemble by $\langle\cdot\rangle$ and using the symbol $\Theta(x)$ to denote the Heaviside step function, one has $I_{\mathcal{S}} \equiv \langle\Theta(\alpha - S)\rangle$. The probability density function, $\mathcal{P}_{\mathcal{S}}(\alpha)$, such that $\mathcal{P}_{\mathcal{S}}(\alpha)\, d\alpha$ gives the probability over the ensemble to find $\alpha < \mathcal{S} \leq \alpha + d\alpha$, is then obtained by differentiation: $\mathcal{P}_{\mathcal{S}}(\alpha) \equiv dI_{\mathcal{S}}/d\alpha = \langle\delta_{\rm D}(\alpha - S)\rangle$ (with $\delta_{\rm D}(x)$ the Dirac delta distribution). The statistical properties of $\mathcal{S}$ are completely determined by the probability density function, for any expectation value can be obtained from $\mathcal{P}_{\mathcal{S}}$ by integration

$$\langle f(\mathcal{S})\rangle = \frac{\int f(\alpha)\,\mathcal{P}_{\mathcal{S}}(\alpha)\, d\alpha}{\int \mathcal{P}_{\mathcal{S}}(\alpha)\, d\alpha} = \int f(\alpha)\,\mathcal{P}_{\mathcal{S}}(\alpha)\, d\alpha \,, \tag{8.1}$$

since, according to our definitions, $\mathcal{P}_{\mathcal{S}}$ is normalized so that

$$\int \mathcal{P}_{\mathcal{S}}(\alpha)\, d\alpha = 1 \,. \tag{8.2}$$

Consider two stochastic variables, $\mathcal{S}_1$ and $\mathcal{S}_2$, their joint probability function can be defined as $\mathcal{P}_{\mathcal{S}_1\mathcal{S}_2} = \langle\delta_{\rm D}(\alpha_1 - \mathcal{S}_1)\,\delta_{\rm D}(\alpha_2 - \mathcal{S}_2)\rangle$. We can also introduce a conditional probability density, $\mathcal{P}_{\mathcal{S}_1|\mathcal{S}_2}(\alpha_1|\alpha_2)$, which is computed by considering only those realizations of the ensemble where $\mathcal{S}_2$ takes the fixed value α_2. In general,

$$\mathcal{P}_{\mathcal{S}_1|\mathcal{S}_2}(\alpha_1|\alpha_2) = \frac{\mathcal{P}_{\mathcal{S}_1\mathcal{S}_2}(\alpha_1, \alpha_2)}{\int \mathcal{P}_{\mathcal{S}_1\mathcal{S}_2}(\alpha_1, \alpha_2)\, d\alpha_1} \,. \tag{8.3}$$

We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are statistically independent if $\mathcal{P}_{\mathcal{S}_1|\mathcal{S}_2}(\alpha_1|\alpha_2) = \mathcal{P}_{\mathcal{S}_1}(\alpha_1)$, that is to say if their joint probability density function $\mathcal{P}_{\mathcal{S}_1\mathcal{S}_2}(\alpha_1, \alpha_2) = \mathcal{P}_{\mathcal{S}_1}(\alpha_1)\,\mathcal{P}_{\mathcal{S}_2}(\alpha_2)$.

The characteristic function of the stochastic variable $\mathcal{S}$ is the average

$$C_{\mathcal{S}}(u) = \langle e^{iu\mathcal{S}}\rangle = \int \mathcal{P}_{\mathcal{S}}(\alpha)\, e^{iu\alpha}\, d\alpha \tag{8.4}$$

which coincides with the Fourier transform of the probability density function. The $n-$th moment μ_n of $\mathcal{S}$, when it exists, can be obtained from the characteristic function by differentiation

$$\mu_n = \langle\mathcal{S}^n\rangle = \frac{1}{i^n}\, \frac{d^n C_{\mathcal{S}}(u)}{du^n}\bigg|_{u=0} \,. \tag{8.5}$$

Hence, the McLaurin expansion of the characteristic function can be written as

$$C_{\mathcal{S}}(u) = 1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!}\mu_n \,. \tag{8.6}$$

The cumulants k_n of $\mathcal{S}$ are instead defined through the relation

$$k_n = \frac{1}{i^n}\, \frac{d^n \ln C_{\mathcal{S}}(u)}{du^n}\bigg|_{u=0} \,, \tag{8.7}$$

which gives

$$\ln C_{\mathcal{S}}(u) = \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} k_n \ . \tag{8.8}$$

The first n cumulants can be expressed in terms of the first n moments and vice versa. For instance, up to $n = 3$: $k_1 = \mu_1$, $k_2 = \mu_2 - \mu_1^2$, $k_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$.

It is clear from equations (8.6) and (8.8) that the whole set of cumulants (or moments) completely determines the probability density function of a stochastic variable. It can be shown (Hanggi & Talkner 1980) that the only probability density functions with a finite number of non-vanishing cumulants are the Dirac delta function ($k_2 = k_3 = \cdots = 0$) and the Gaussian ($k_3 = k_4 = \cdots = 0$).

8.1.2 Random fields

Roughly speaking, a random field is a infinite and continuous collection of random variables each of which is associated with a point of some space. For instance, the temperature distribution in a room or the elevation of the ground above sea level are random fields (the first one is defined within a cuboid while the second is defined on the surface of a sphere).

In order to give a more rigourous mathematical definition, let us consider a space of functions (or, more generally, distributions), $\mathcal{F}$, characterized by some precise requirements (e.g. functions having a Lebesgue-integrable square modulus over a specified domain) and endowed with a scalar product and a norm. A n-dimensional random field, $\mathcal{R}(\mathbf{x})$, is a set of random variables, one for each point in a n-dimensional real space, $\mathcal{T}$ such that $\mathbf{x} \in \mathcal{T}$, defined by a probability functional, $\mathcal{P}[\hat{\mathcal{R}}(\mathbf{x})]$, which specifies the probability for the occurrence of a particular realization of the field (i.e. of the function $\hat{\mathcal{R}}(\mathbf{x}) \in \mathcal{F}$) over the ensemble. Such a probability distribution is normalized by requiring that

$$\int D[\mathcal{R}(\mathbf{x})]\mathcal{P}[\mathcal{R}(\mathbf{x})] = 1 \ , \tag{8.9}$$

where $\int D[\mathcal{R}(\mathbf{x})]$ denotes functional integration over $\mathcal{F}$ with measure $\int D[\mathcal{R}]$.

In analogy with equation (8.4), we can introduce the partition functional

$$\mathcal{Z}[J(\mathbf{x})] = \int D[\mathcal{R}(\mathbf{x})]\, \mathcal{P}[\mathcal{R}(\mathbf{x})]\, \exp\left[i \int d^3x\, \mathcal{R}(\mathbf{x})\, J(\mathbf{x}) \right] \ , \tag{8.10}$$

and define the disconnected correlation functions as the coefficients of its McLaurin expansion

$$\mathcal{Z}[J(\mathbf{x})] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^3x_1 \ldots \int d^3x_n\, \xi_n^{(\mathrm{dis})}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, J(\mathbf{x}_1) \ldots J(\mathbf{x}_n) \ . \tag{8.11}$$

Similarly, we define the connected correlation functions are the coefficients of

$$\ln \mathcal{Z}[J(\mathbf{x})] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^3x_1 \ldots \int d^3x_n\, \xi_n^{(\mathrm{con})}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, J(\mathbf{x}_1) \ldots J(\mathbf{x}_n) \ . \tag{8.12}$$

One nice property of the connected functions is that they vanish as any subset of $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is removed to infinite separation.

As a probability distribution function is fully specified by the entire hierarchy of its moments or cumulants, a random field is fully specified by the entire hierarchy of its "correlation functions" (either connected or disconnected). Note that the disconnected ones can be written as

$$\xi_n^{(\mathrm{dis})}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \langle \mathcal{R}(\mathbf{x}_1), \ldots, \mathcal{R}(\mathbf{x}_n) \rangle \ . \tag{8.13}$$

with n an integer dubbed the "order" of the correlation function. These functions measure the spatial coherence of a random field.

For a random field with a vanishing mean value, the two-point function,

$$\langle \mathcal{R}(\mathbf{x}_1)\, \mathcal{R}(\mathbf{x}_2) \rangle = \xi^{(\mathrm{dis})}(\mathbf{x}_1, \mathbf{x}_2) = \xi^{(\mathrm{con})}(\mathbf{x}_1, \mathbf{x}_2) \ , \tag{8.14}$$

is called the auto-correlation function of the field. From now on, we will use the symbol ξ to indicate this function.

8.1.3 Homogeneity and isotropy

Random fields are used in cosmology to describe fluctuations in the distribution of a number of different quantities with respect to a smooth background. The typical example is the three-dimensional mass overdensity field:

$$\delta(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t) - \bar{\rho}(t)}{\bar{\rho}(t)} \ , \tag{8.15}$$

where $\bar{\rho}(t)$ denotes the mean density of the underlying background FRW model. In simple words, $\delta(\mathbf{x}, t)$ gives the fractional overdensity at a given epoch and in a given region of space with respect to the unperturbed case.

From now on we will only consider three-dimensional random fields defined on a perturbed hypersurface of homogeneity at constant cosmic time. Not to violate the cosmological principle (which is one the fundamental basis of our theory), we only want to consider stochastic fields which are statistically homogeneous and isotropic. A random field is homogeneous if all its connected correlation functions are invariant under simultaneous translations of their arguments by the same vector. For instance, this implies that $\langle \mathcal{R}(\mathbf{x}) \rangle$ is spatially independent and $\xi(\mathbf{x}_1, \mathbf{x}_2)$ is a function only of the relative separation $\mathbf{x}_{12} = \mathbf{x}_2 - \mathbf{x}_1$. Isotropy instead means invariance of the correlation functions under a global rotation of their arguments. In this case ξ depends only on $x_{12} = |\mathbf{x}_{12}|$. A random field which is homogeneous and isotropic is generally called "stationary".

8.1.4 Fourier transform

It is often convenient to consider a random field as the linear superposition of "simple" waves. In a spatially flat Universe, this is generally achieved via Fourier analysis. For non-flat models, plane waves are not a complete set and should be replaced

by the eigenfunctions of the corresponding wave equation. It is clear, however, that the difference only matters on scales which are comparable with the curvature radius of the Universe and thus on the Hubble length. Since we will only discuss structure formation in the Newtonian framework (which is only valid on scales that are much smaller than the Hubble length), we can safely ignore this complication.

Our convention for the Fourier transform of the field $\mathcal{R}(\mathbf{x}, t)$ is

$$\widetilde{\mathcal{R}}(\mathbf{k}, t) = \int \mathcal{R}(\mathbf{x}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x \,, \tag{8.16}$$

$$\mathcal{R}(\mathbf{x}, t) = \int \widetilde{\mathcal{R}}(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \frac{d^3k}{(2\pi)^3} \,. \tag{8.17}$$

8.1.5 The Wiener-Khintchine theorem

Consider the expectation value for the product of two Fourier modes of a three-dimensional random field

$$\langle \widetilde{\mathcal{R}}(\mathbf{k})\, \widetilde{\mathcal{R}}^*(\mathbf{k}') \rangle = \langle \int \mathcal{R}(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x\, \mathcal{R}(\mathbf{y})\, e^{-i\mathbf{k}'\cdot\mathbf{y}}\, d^3y \rangle \,. \tag{8.18}$$

Since the only stochastic variables are the random fields and all the operations involving them are linear, we can exchange the order between the spatial integrations and the ensemble average thus obtaining

$$\langle \widetilde{\mathcal{R}}(\mathbf{k})\, \widetilde{\mathcal{R}}^*(\mathbf{k}') \rangle = \int \langle \mathcal{R}(\mathbf{x})\, \mathcal{R}(\mathbf{y}) \rangle\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x\; e^{-i\mathbf{k}'\cdot\mathbf{y}}\, d^3y \,. \tag{8.19}$$

The ensemble average gives the two-point correlation function so that

$$\langle \widetilde{\mathcal{R}}(\mathbf{k})\, \widetilde{\mathcal{R}}^*(\mathbf{k}') \rangle = \int \xi(\mathbf{y} - \mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x\; e^{-i\mathbf{k}'\cdot\mathbf{y}}\, d^3y \,. \tag{8.20}$$

It is convenient to replace the spatial variables $\mathbf{x}$ and $\mathbf{y}$ with the new set $\mathbf{x}$ and $\mathbf{r} = \mathbf{y} - \mathbf{x}$ which, since the Jacobian determinant of the transformation is -1, gives

$$\langle \widetilde{\mathcal{R}}(\mathbf{k})\, \widetilde{\mathcal{R}}^*(\mathbf{k}') \rangle = \int \xi(\mathbf{r})\, e^{-i\mathbf{k}'\cdot\mathbf{r}}\, d^3r \int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\, d^3x \,. \tag{8.21}$$

Since,

$$\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\, d^3x = (2\pi)^3 \delta_{\rm D}(\mathbf{k} - \mathbf{k}') \,, \tag{8.22}$$

we finally obtain

$$\langle \widetilde{\mathcal{R}}(\mathbf{k})\, \widetilde{\mathcal{R}}^*(\mathbf{k}') \rangle = (2\pi)^3 \delta_{\rm D}(\mathbf{k} - \mathbf{k}')\, P(\mathbf{k}) \tag{8.23}$$

with

$$P(\mathbf{k}) = \int \xi(\mathbf{r})\, e^{-i\mathbf{k}'\cdot\mathbf{r}}\, d^3r \tag{8.24}$$

the power spectrum of the random field $\mathcal{R}$. This is known as Wiener-Khintchine theorem. In particular, if the random field assumes real values, one has $\widetilde{\mathcal{R}}^*(\mathbf{k}) = \widetilde{\mathcal{R}}(-\mathbf{k})$ which gives

$$\langle \widetilde{\mathcal{R}}(\mathbf{k}) \widetilde{\mathcal{R}}(\mathbf{k}') \rangle = (2\pi)^3 \delta_{\rm D}(\mathbf{k} + \mathbf{k}')\, P(\mathbf{k}) \; . \tag{8.25}$$

Spatial homogeneity and isotropy imply that the power spectrum, $P(\mathbf{k})$, is a function of $k = |\mathbf{k}|$. This is easily shown as follows. Take eq. (8.24) and consider spherical coordinates (r, θ, φ) such that the angle θ measures the polar angle with respect to the vector $\mathbf{k}$. This gives:

$$P(\mathbf{k}) = \int \xi(r)\, e^{-ikr\cos\theta}\, r^2\, dr\, d\cos\theta\, d\varphi \; . \tag{8.26}$$

Now,

$$\begin{aligned} \int_0^{2\pi} d\varphi &= 2\pi \\ \int_{-1}^{1} e^{-ikr\cos\theta}\, d\cos\theta &= \frac{e^{ikr} - e^{-ikr}}{ikr} = 2\,\frac{\sin(kr)}{kr} \; , \end{aligned}$$

which implies

$$P(\mathbf{k}) = P(k) = 4\pi \int_0^\infty r^2\, \xi(r)\, \frac{\sin(kr)}{kr}\, dr \; . \tag{8.27}$$

8.1.6 Correlation and power spectrum for a stationary random field

The inverse transformation which gives the two-point correlation function for a given power spectrum of a stationary three-dimensional field is easily derived starting from the definition of $\xi(r)$.

$$\begin{aligned} \xi(r) &= \langle \mathcal{R}(\mathbf{x})\, \mathcal{R}(\mathbf{x}+\mathbf{r}) \rangle = \\ &= \langle \int \widetilde{\mathcal{R}}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \widetilde{\mathcal{R}}(\mathbf{k}')\, e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})}\, \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} = \\ &= \int \langle \widetilde{\mathcal{R}}(\mathbf{k})\, \widetilde{\mathcal{R}}(\mathbf{k}') \rangle e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})}\, \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} = \\ &= \int (2\pi)^3\, \delta_{\rm D}(\mathbf{k} + \mathbf{k}')\, P(k)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})}\, \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} = \\ &= \int P(k)\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{d^3k}{(2\pi)^3} = \\ &= \frac{1}{2\pi^2} \int_0^\infty k^2\, P(k) \frac{\sin(kr)}{kr}\, dk \; . \end{aligned} \tag{8.28}$$

8.1.7 The variance

In the limit for $r \to 0$, the two-point correlation function reduces to the variance

$$\sigma^2 = \langle \mathcal{R}^2(\mathbf{x}) \rangle \ . \tag{8.29}$$

Since $\lim_{r\to 0} \sin(kr)/(kr) = 1$, eq. (8.28) gives

$$\sigma^2 = \frac{1}{2\pi^2} \int_0^\infty k^2 \, P(k) \, dk \ . \tag{8.30}$$

This equation gives a more easily understandable meaning to the power spectrum than its definition. The product $P(k)/(2\pi)^3$ gives the power (i.e. the contribution to the variance of the field) per unit k-space volume due to the modes with wavevector k. In three-dimensional space, however, a thin spherical shell in k-space contains many different modes and has a k-space volume $4\pi \, k^2 \, dk$. Therefore, the total "power" contributed by fluctuations with wavenumber between k and $k + dk$ is proportional to the product of the power spectrum and the "number" (i.e. the k-space volume) of contributing modes.

It is convenient to introduce a new quantity (which is often called the "dimensionless power spectrum")

$$\Delta^2(k) = \frac{d\sigma^2}{d \ln k} = \frac{1}{2\pi^2} \, k^3 \, P(k) \tag{8.31}$$

which gives the contribution to the variance of the random field per bin of $\ln k$. When mass-density fluctuations are considered, saying that $\Delta^2(k) = 1$ means that the Fourier modes in a unit logarithmic bin around wavenumber k generate fluctuations $\delta\rho/\rho$ of order unity.

8.1.8 Gaussian random fields

A Gaussian scalar random field, $G(\mathbf{x})$, with vanishing mean value is defined by the probability functional

$$\mathcal{P}[G] = (\det K)^{1/2} \, \exp\left\{ -\frac{1}{2} \int G(\mathbf{x}) \, K(\mathbf{x}, \mathbf{y}) \, G(\mathbf{y}) \, d^3x \, d^3y \right\} \ , \tag{8.32}$$

where $K(\mathbf{x}, \mathbf{y})$ is an invertible operator, symmetric with respect to the variables $\mathbf{x}$ and $\mathbf{y}$. This is the functional inverse of the two-point correlation function. i.e.

$$\int K(\mathbf{x}, \mathbf{y}) \, K^{-1}(\mathbf{y}, \mathbf{z}) \, d^3y = \delta_{\rm D}(\mathbf{x} - \mathbf{z}) \tag{8.33}$$

with $\xi(\mathbf{x}, \mathbf{y}) = K^{-1}(\mathbf{x}, \mathbf{y})$. Equation (8.12) then gives $\xi_n^{(\rm con)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$ for $n > 2$. Thus, a Gaussian random field with vanishing mean value is completely determined by its two-point connected function.

In practice, a Gaussian random field is a collection of stochastic variables such that the joint probability function of the values of $G(\mathbf{x})$ in the points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a multivariate Gaussian distribution

$$\mathcal{P}_n(\alpha_1, \ldots, \alpha_n) d\alpha_1 \ldots d\alpha_n = [(2\pi)^n \det M]^{-1/2} \exp\left[-\frac{1}{2}\alpha_i M^{-1}_{ij} \alpha_j\right] d\alpha_1 \ldots d\alpha_n \,, \tag{8.34}$$

with $M_{ij} = \langle \alpha_1 \alpha_j \rangle = \xi(x_{ij})$ the covariance matrix. Note that this equation is simply the discrete version of eq. (8.32). Also note that the field and its derivatives are jointly Gaussian.

Rigorously, in order for a field to be homogeneous and Gaussian, its Fourier modes, $\widetilde{G}(\mathbf{k}) = |\widetilde{G}(\mathbf{k})| \, e^{i\theta(\mathbf{k})}$, must be statistically independent, have random phases and have moduli which are Rayleigh distributed. For a given wavevector $\mathbf{k}$, this is equivalent to

$$\mathcal{P}[|\widetilde{G}(\mathbf{k})|, \theta(\mathbf{k})] = \exp\left[-\frac{|\widetilde{G}(\mathbf{k})|^2}{P(k)}\right] \frac{2\,|\widetilde{G}(\mathbf{k})|^2}{P(k)} \, d|\widetilde{G}(\mathbf{k})| \, \frac{d\theta(\mathbf{k})}{2\pi} \,, \tag{8.35}$$

where $P(k)$ is the power spectrum of G. The real and imaginary part of $\widetilde{G}(\mathbf{k})$ are then mutually independent and Gaussian distributed with variance $P(k)/2$.

Loosely, any sum of a large number of independent modes with random phases will tend to a Gaussian random field. Since quantum field theory predicts that different modes of a free scalar field are independent and sum up incoherently, we expect that the perturbations generated by the zero-point fluctuations of the inflaton produce a Gaussian random field. Note that even before the advent of inflationary theories for simplicity it was almost always assumed that primordial fluctuations were Gaussian.

8.1.9 Ergodicity and fair samples

In cosmology, when we assume that fluctuations in the energy density (or in the gravitational potential) at some primordial epoch (e.g. after the end of the inflationary phase) form a three-dimensional random field, we mean that they represent one particular realization extracted from an ideal ensemble of functions. Theoretical investigation is then only able to compute statistical quantities averaged over this ensemble. On the other hand, since astronomical observations refer to a single realization of the random field, in the observable universe we can only take spatial averages over large volumes. Therefore, theoretical results can be tested against observational data only if the primordial fluctuations are ergodic, i.e. if spatial averages over large volumes in a single realization are equal to expectations over the ensemble. A Gaussian random field can be shown to be ergodic if and only if its power spectrum is continuos. In general, the validity of the ergodic hypothesis depends on the ratio between the length scale over which we perform the spatial average and the scale at which spatial correlations become negligibly small. When the observationally surveyed volume contains many statistically independent subsamples (i.e. it is a fair sample of the Universe), ergodicity is expected to hold.

8.2 Coarse graining

Astronomical observations and N-body simulations deal with discrete units (e.g. galaxies or particles) instead of a continuum. Smoothing over a finite volume is therefore required in order to compare data and theoretical results based on a "fluid-like" description of the cosmological mass distribution. It is then of extreme importance to study the statistical properties of the density fluctuation field at some finite resolution scale $R_{\rm f}$. This is introduced by convolving the random field $\delta(\mathbf{x}, t)$ by some filter function $W(|\mathbf{x}' - \mathbf{x}|, R_{\rm f})$,

$$\delta(\mathbf{x}, t; R_{\rm f}) = \int W(|\mathbf{x}' - \mathbf{x}|, R_{\rm f})\, \delta(\mathbf{x}', t)\, d^3x' = \int \widetilde{W}(kR_{\rm f})\, \tilde{\delta}(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3k}{(2\pi)^3}\ , \quad (8.36)$$

where $\widetilde{W}$ is the Fourier transform of the filter and is generally called the "window function". At every point $\mathbf{x}$, the smoothed field represents the weighted average of δ over a spherical region of characteristic dimension $R_{\rm f}$ centred in $\mathbf{x}$. The most commoly used smoothing kernels are the top-hat filter

$$W_{\rm TH}(x, R_{\rm f}) = \frac{3}{4\pi R_{\rm f}^3}\, \Theta(R_{\rm f} - x) \quad (8.37)$$

(where $\Theta(x)$ is the Heaviside step function), and the Gaussian one

$$W_{\rm G}(x, R_{\rm f}) = \frac{1}{(2\pi R_{\rm f}^2)^{3/2}}\, \exp\left(\frac{-x^2}{2R_{\rm f}^2}\right)\ . \quad (8.38)$$

The corresponding window functions are

$$\widetilde{W}_{\rm TH}(y) = 3\frac{\sin(y) - y\cos(y)}{y^3} \quad (8.39)$$

and

$$\widetilde{W}_{\rm G}(y) = \exp\left(-\frac{y^2}{2}\right) \quad (8.40)$$

with $y = kR_{\rm f}$.

It is trivial to show that the power spectrum of the coarse-grained density field is given by the product $\widetilde{W}^2(kR_{\rm f})\, P(k, t)$, so that the corresponding variance is

$$\sigma^2(t, R_{\rm f}) \equiv \langle \delta^2(\mathbf{x}, r; R_{\rm f}) \rangle = \frac{1}{2\pi^2} \int k^2 \widetilde{W}^2(kR_{\rm f})\, P(k, t)\, dk\ . \quad (8.41)$$

8.2.1 An example: scale-invariant spectra

In the absence of a physical theory for the generation of density fluctuations, a simple model for the power spectrum of density fluctuation which does not include any characteristic scale is

$$P(k) = A\, k^n \quad (8.42)$$

where the *spectral index n* governs the balance between large- and small-scale power and A is a normalization constant.

In this case,

$$\sigma^2(R_{\rm f}) = \frac{A}{2\pi^2}\int k^{n+2}\,\widetilde{W}^2(kR_{\rm f})\,dk\ . \tag{8.43}$$

Introducing the new variable, $y = kR_{\rm f}$

$$\sigma^2(R_{\rm f}) = \frac{A}{2\pi^2}\,\frac{1}{R_{\rm f}^{n+3}}\int y^{n+2}\,\widetilde{W}^2(y)\,dy\ . \tag{8.44}$$

We can associate a characteristic mass to each value of the smoothing radius by computing the mean mass contained within the filter:

$$M(R_{\rm f}) = \bar{\rho}\int W(x;R_{\rm f})\,d^3x \propto R_{\rm f}^3\ . \tag{8.45}$$

We can then express the variance of the smoothed field as a function of this mass

$$\sigma^2(M) \propto M^{-(n+3)/3}\ . \tag{8.46}$$

Note that in order to guarantee large-scale homogeneity the spectral index has to be smaller than -3 (otherwise fluctuations do not decrease when considering larger scales). It can also be shown that it is impossible to create spectra with $n \geq 4$ from a discrete distribution of matter. Therefore, the range of variability for the spectral index is

$$-3 < n < 4\ . \tag{8.47}$$

A particular case is that of "white noise" ($n = 0$) (the same power at all wavelengths), where one gets Poissonian fluctuations $\sigma(M) \propto M^{-1/2}$. This is the case which corresponds to throwing down in space a large number of massive particles at random.

As we will discuss in detail, the simplest model of inflation tend to produce a spectrum with $n \simeq 1$ (Harrison-Zel'dovich spectrum), which corresponds to $\sigma(M) \propto M^{-2/3}$.

Even though the characteristic spectrum of density perturbations is not a simple power-law, it makes sense to define an effective spectral index such that:

$$n_{\rm eff}(k) = \frac{d\ln P(k)}{d\ln k}\ . \tag{8.48}$$

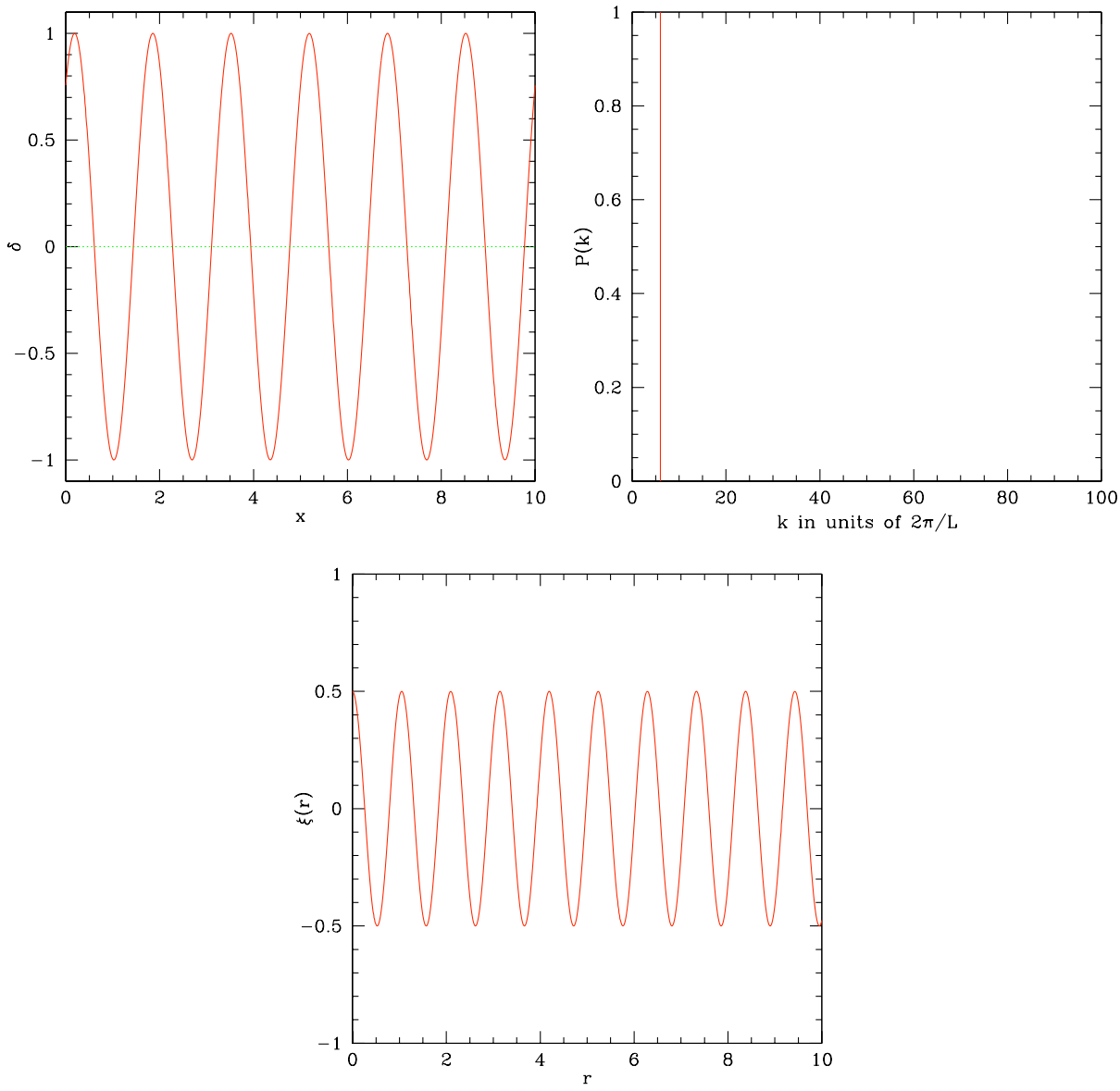

Figure 8.1: A one-dimensional random field containing a single Fourier mode (top-left), its power spectrum (top-right), and its two-point correlation function (bottom).

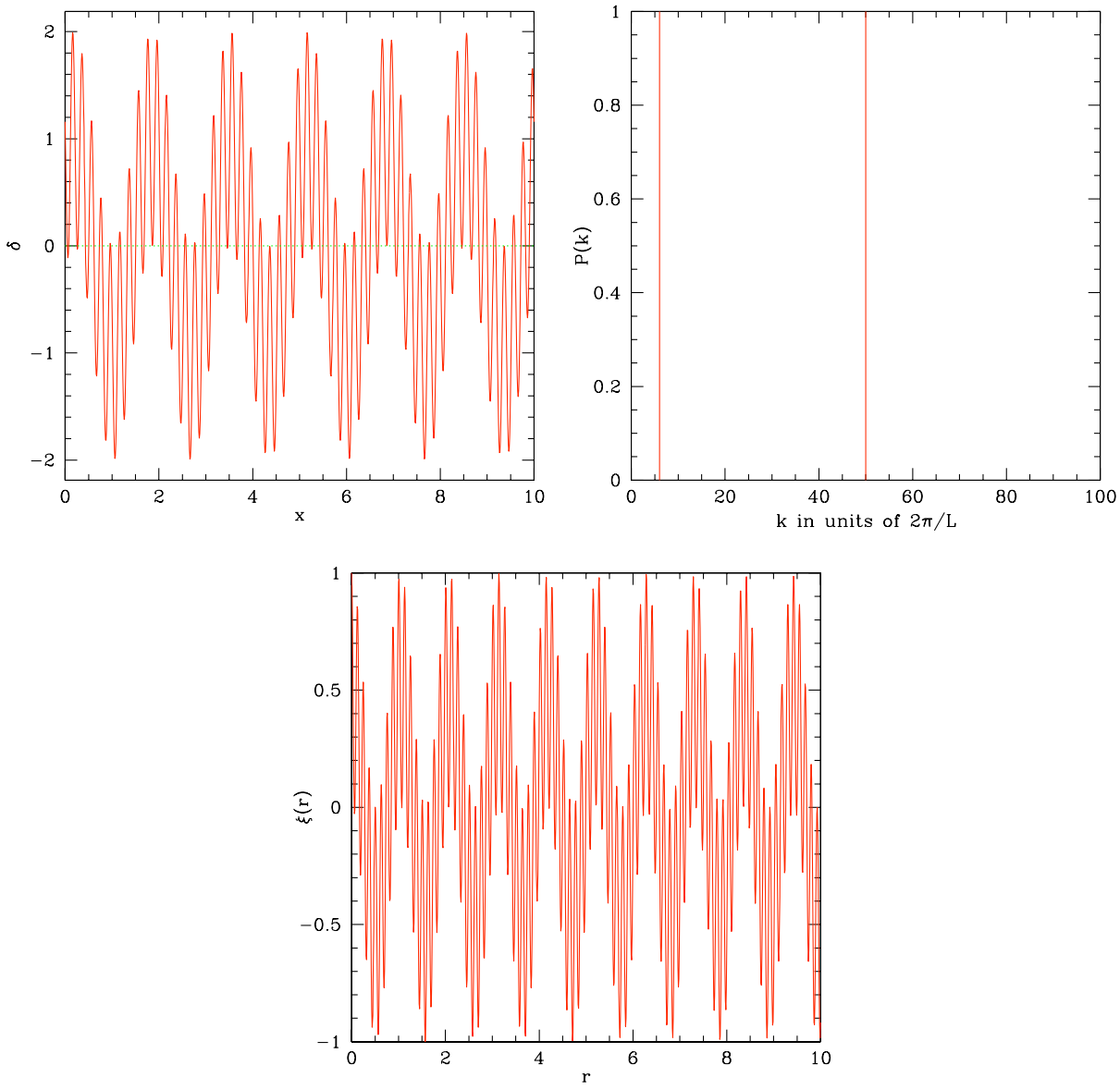

Figure 8.2: A one-dimensional random field obtained summing up two sine-functions with different frequencies and amplitudes (top-left), its power spectrum (top-right), and its two-point correlation function (bottom).

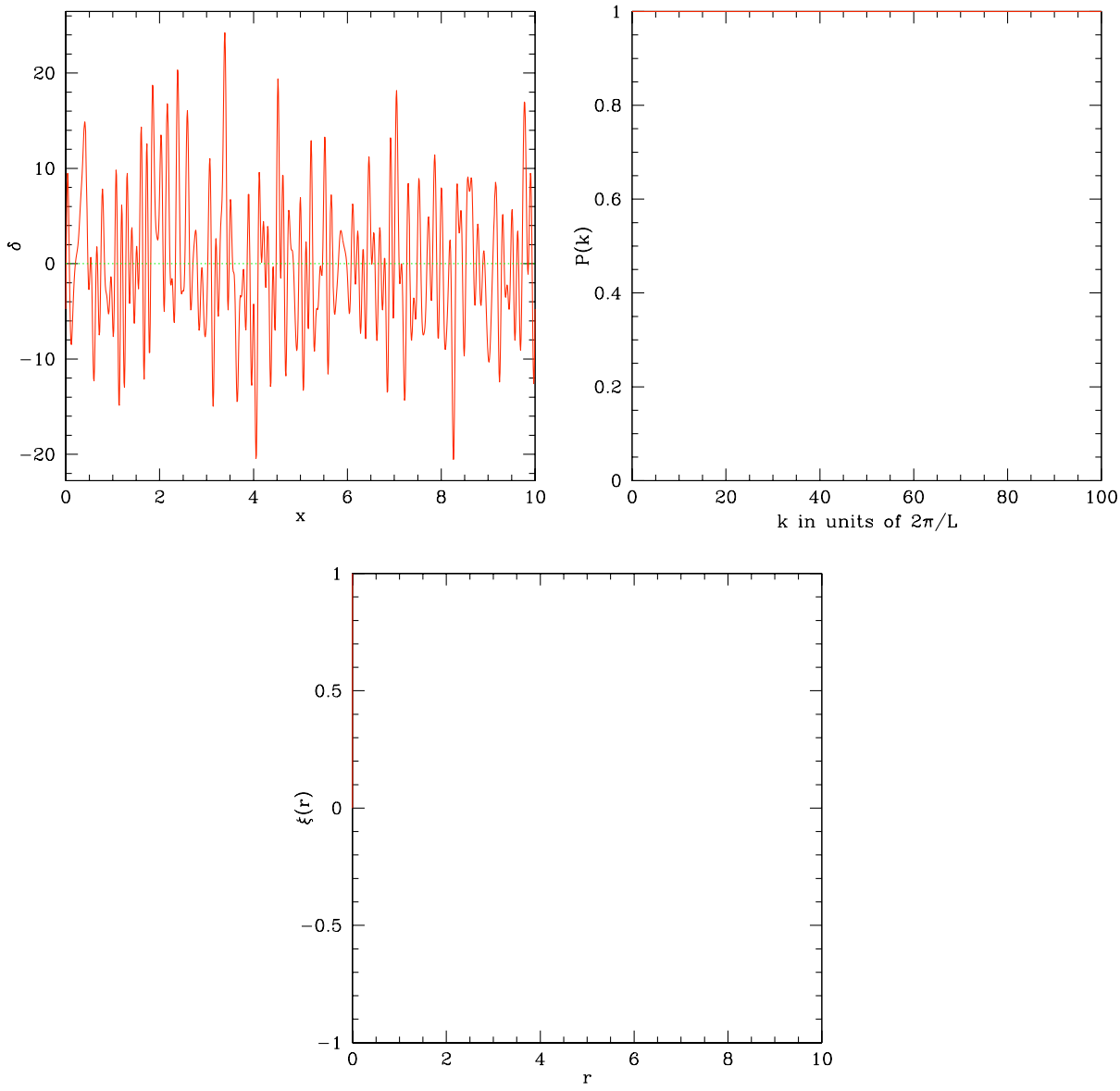

Figure 8.3: A one-dimensional random field with a white-noise spectrum (top-left), its power spectrum (top-right, $P(k) = 1$), and its two-point correlation function (bottom, $\xi(r) \propto \delta_{\rm D}(r)$).

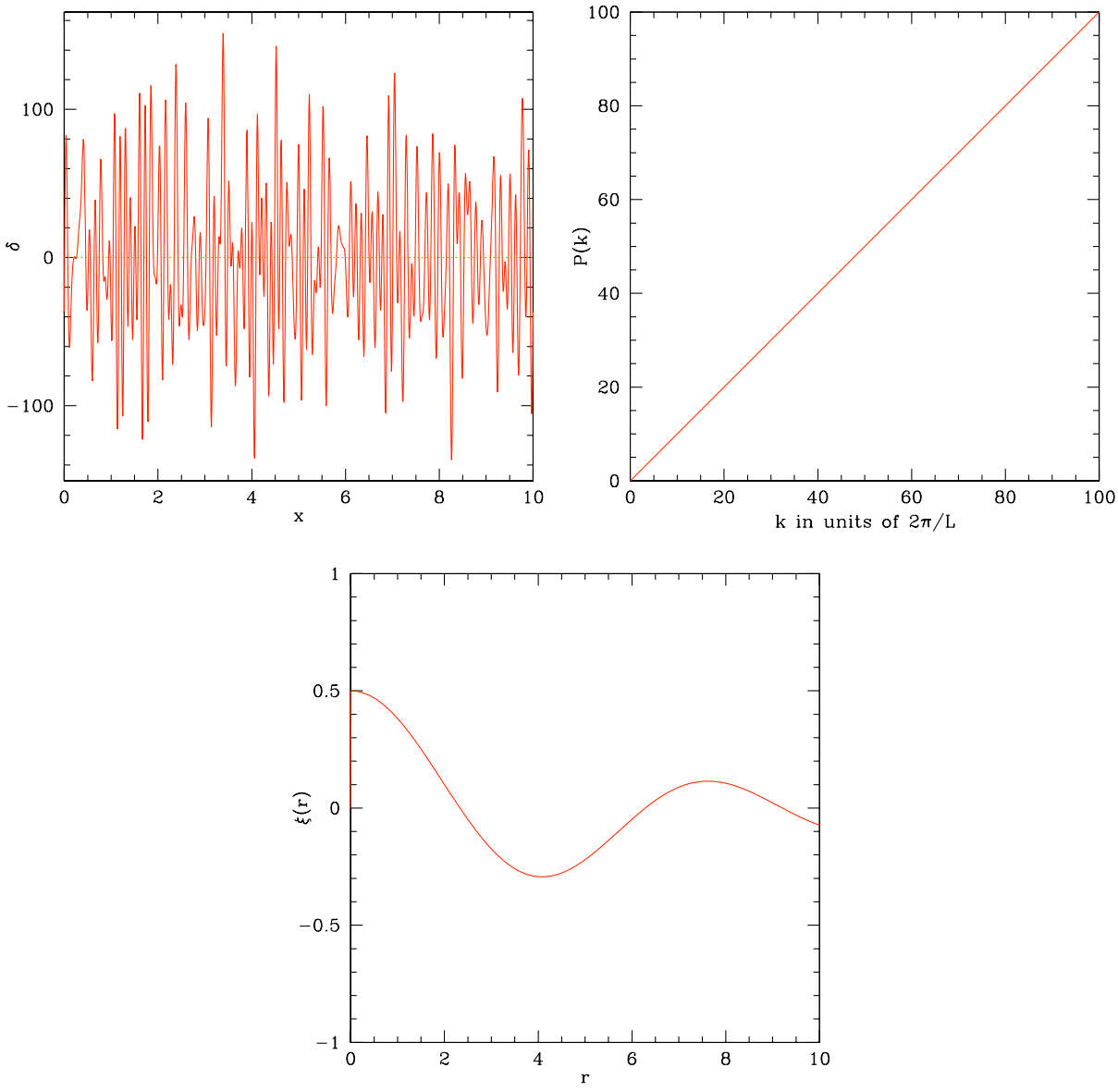

Figure 8.4: A one-dimensional random field with a power spectrum $P(k) = k$ (top-left), its power spectrum (top-right), and its two-point correlation function (bottom).

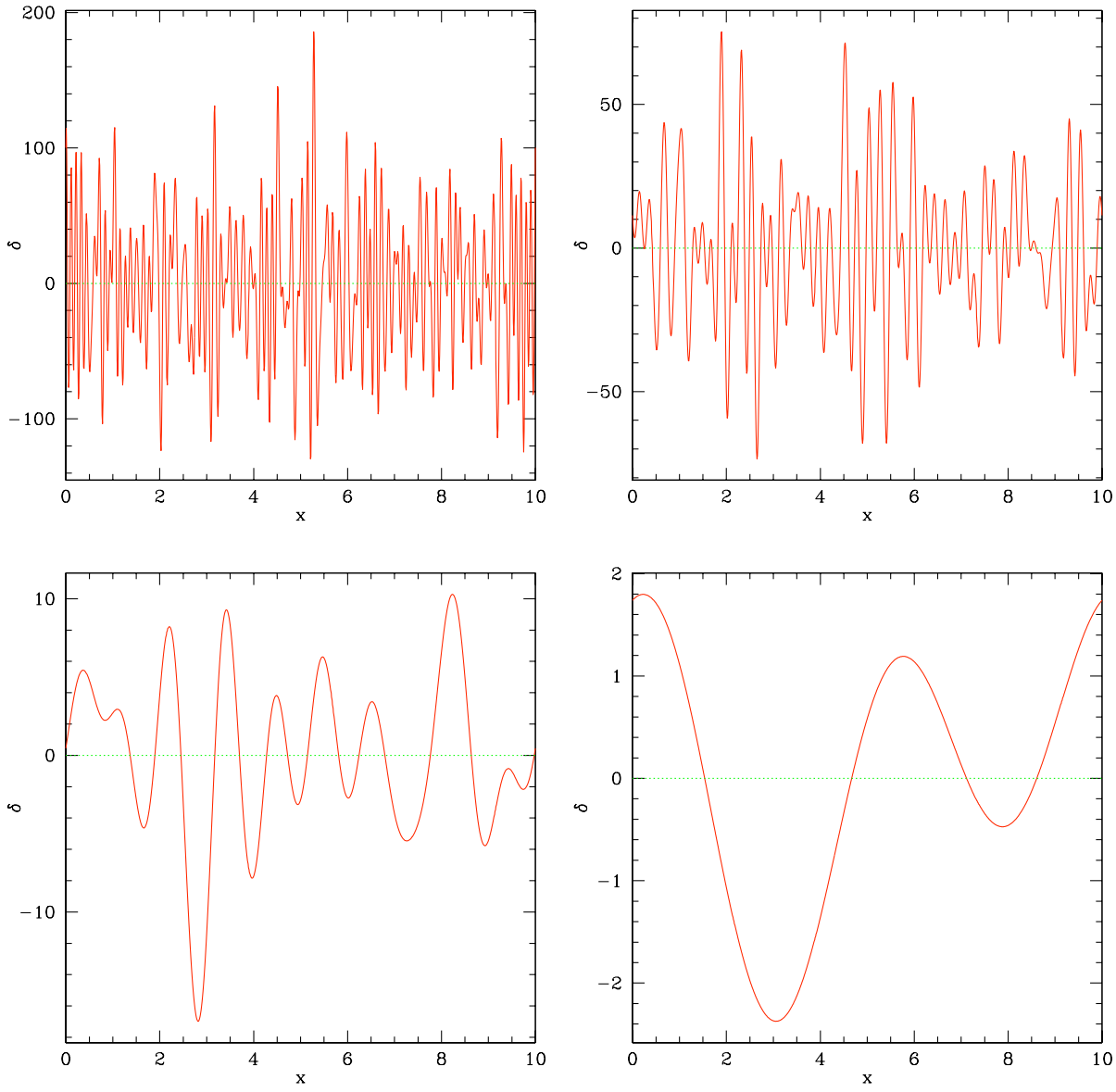

Figure 8.5: A (one-dimensional) random field is observed at different spatial resolutions. The highest resolution (top-left panel) is obtained using a smoothing length of $R_{\rm f} = 0.1$ and corresponds to a rms value for δ of $\sigma_\delta = \langle\delta^2\rangle = 50.3$. The top-right, bottom-left and bottom-right panels respectively correspond to $R_{\rm f} = 0.2, 1.0, 1.5$ and to $\sigma_\delta = 25.2, 5.2$ and 1.2. Periodic boundary conditions have been adopted everywhere.