School of Astrophysics "Francesco Lucchin"

Three Lectures on
Long-term Evolution of Globular Clusters

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Course outline (tentative)

Lecture 1: Fundamental aspects of half-mass radius evolution.

- References.
- The globular clusters of the Milky Way
- Evolution of the half-mass radius
- Effects of the Galactic tide

Lecture 2: Fundamental aspects of core radius evolution

- The core radius
- Energy generation by stellar evolution
- Hénon’s Principle
- The dynamical role of binary stars
- Isothermal models and their stability

Lecture 3: Modelling globular clusters and their long-term evolution

- Dynamical evolutionary models: *N*-body methods
- Monte Carlo method
- A case study: M4

References

1. “Galactic Dynamics”, 2e James Binney and Scott Tremaine, Princeton University Press, 2008, 885 pp.
2. “Dynamical Evolution of Globular Clusters” Lyman J. Spitzer Jr, Princeton UP, 1988, 196 pp.
3. “The Gravitational Million Body Problem”, Douglas Heggie, Piet Hut; Cambridge UP, 2003
4. “The Cambridge *N*-body Lectures, S.J. Aarseth, C.A. Tout, R.A. Mardling, eds. Springer, LNP760, 2008, 402pp

Lecture 1: Fundamental aspects of half-mass radius evolution

Figure: 47 Tuc (image credit T.V. Davis)

The globular clusters of the Milky Way

- 157 listed at http://physwww.physics.mcmaster.ca/~harris/mwgc.dat
- Typical radius: a few pc
- Distance from Galactic Centre 0.5 – 125 kpc
- Almost all have ages $\sim 12 Gyr$
- Typical internal velocities: a few – 20 km/s (Mclaughlin & van der Marel, 2005, ApJS, 161, 304)
- Typical $M/L_V \simeq 1.45 M_\odot/L_\odot$ (McLaughlin, 2000, ApJ, 539, 618)
- Median mass $\sim 3 \times 10^5 M_\odot$
- Fraction of binaries in the “core” almost all $\lesssim 10\%$, with outliers (Milone et al, 2008, MmSAI, 79, 623)
- Low rotation speed and ellipticity (White & Shawl, 1987, ApJ, 317, 246)

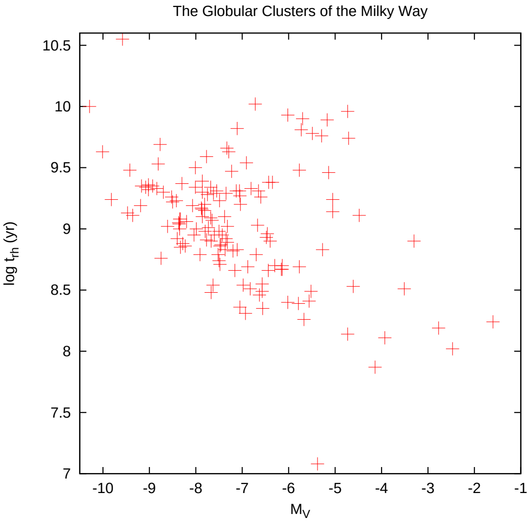
Why globular clusters are important

- Size of the Galaxy (Shapley 1918)
- Age of the universe (Sandage 1950s)
- Accretion onto the Galaxy (e.g. clusters associated with the Sagittarius dwarf)
- Stellar evolution
- Stellar initial mass function and binarity
- Stellar collisions - blue stragglers (lectures by M. Davies, F. Ferraro)
- High-energy astrophysics: X-ray binaries, millisecond pulsars (lectures by M. Davies, F. Ferraro)

The two time scales of star cluster dynamics

1. The crossing time t_{cr}
   - $\sim$ 1Myr for the globular star clusters of the Milky Way
   - $\sim$ “period” of orbital motions of stars
   - $\sim$ time scale of virialisation
2. The relaxation time t_{rh}
   - $\sim \dfrac{N}{\log N} t_{cr}$
   - $\gg t_{cr}$
   - A fraction of the age of the universe for the globular clusters
   - The time scale of evolution of a virialised cluster
   - Escape rate is $\sim 100 t_{rh}$[1]

[1] Binney & Tremaine, p.556



Data: http://www.physics.mcmaster.ca/~harris/mwgc.dat

The three length scales of a globular cluster

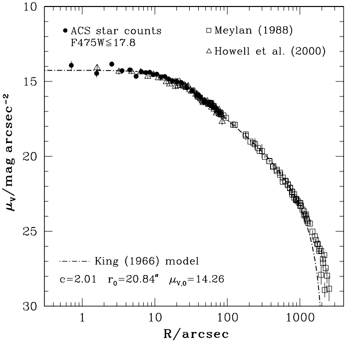
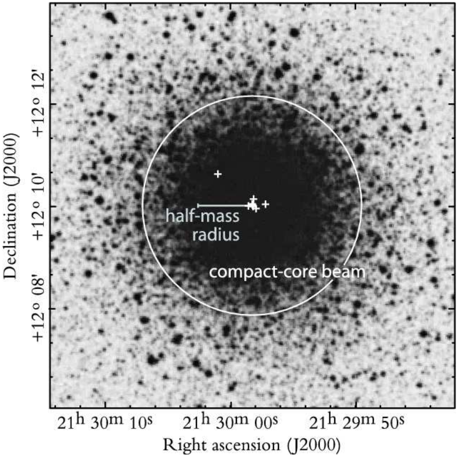


Figure: Surface brightness profile of 47 Tuc (McLaughlin et al, 2006)

- Core radius $r_c = 22''$
- Half-mass radius $r_h = 190''$ (half-light radius)
- Tidal radius $r_t = 2500''$

The half-mass radius

Definition: The radius of a sphere, centred at the centre of the cluster, which encloses half of the mass of the cluster.



Example: M15 (van Leeuwen & Stappers, 2010, A&A, 509, A7)

The star-star cluster analogy

Stars	**Star Clusters**
Temperature decreases with radius	Velocity dispersion decreases with radius
Outward flow of energy (radiative or convective) from hot to cold	Outward flow of energy (via two-body encounters) from fast to slow-moving stars
Central energy source (thermonuclear)	Central energy source (see Lecture 2)
Surface flux of energy (radiation)	Surface is nearly insulated (mass-loss too slow: 2)
Equilibrium (central source balances flux at surface)	**Expansion** Energy $E \sim -\frac{GM}{r_h}$ $\dot{E} > 0, \dot{M} \simeq 0 \Rightarrow r_h$ increases

Expansion of the half-mass radius

Expansion of the half-mass radius caused by a flow of energy, which is caused by two-body relaxation, with time scale t_{rh}. Hence

$$\frac{\dot{r}_h}{r_h} = \frac{\zeta}{t_{rh}},$$

where $\zeta \simeq$ constant. (Actually ζ depends on the stellar mass function.) Also

$$t_{rh} = 0.138\frac{N^{1/2} r_h^{3/2}}{m^{1/2} G^{1/2} \ln\Lambda},$$

(Spitzer 1987, p.40), where

- N = number of stars
- m = (average) mass
- G = gravitational constant
- $\Lambda \simeq \lambda N$, where λ is constant.

See also talk by Poul Alexander

Expansion of the half-mass radius (continued)

If we neglect escape, $t_{rh} \propto r_h^{3/2}$, and so

$$\frac{t_{rh}(t)}{t_{rh}(0)} \simeq \left(\frac{r_h(t)}{r_h(0)}\right)^{3/2},$$

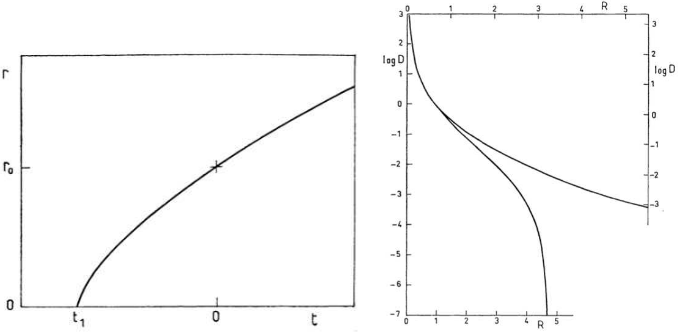
where $t = 0$ is some arbitrary reference time. Since $\dot{r}_h/r_h = \zeta/t_{rh}(t)$,

$$\begin{aligned}
\frac{\dot{r}_h}{r_h} &= \frac{\zeta}{t_{rh}(0)}\left(\frac{r_h(0)}{r_h}\right)^{3/2} \\
\Rightarrow r_h^{1/2}\dot{r}_h &= \frac{\zeta}{t_{rh}(0)} r_h(0)^{3/2} \\
\Rightarrow \frac{2}{3}\left(r_h^{3/2} - r_h(0)^{3/2}\right) &= \frac{\zeta}{t_{rh}(0)} r_h(0)^{3/2} t \\
\Rightarrow r_h(t) &= r_h(0)\left(1 + \frac{3}{2}\frac{\zeta}{t_{rh}(0)} t\right)^{2/3} \qquad (1) \\
\Rightarrow t_{rh}(t) &= t_{rh}(0)\left(1 + \frac{3}{2}\frac{\zeta}{t_{rh}(0)} t\right) \qquad (2)
\end{aligned}$$

Hénon’s model

A numerical solution of the Fokker-Planck equation.

M. Henon, 1965, *Sur l’évolution dynamique des amas globulaires. II. Amas isolé*, Annales d’Astrophysique, **28**, 62 (transl. by F. Renaud in arXiv:1103.3498).



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(a) Evolution of r_h

(b) Density profile (upper curve)

N-body models showing expansion

Baumgardt *et al*, 2002, MNRAS, **336**, 1069

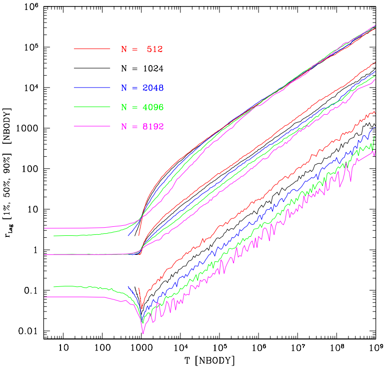


Figure: r_h (middle series of curves) in systems with N = 512–8192 equal-mass stars

In the early evolution there is no balance between the energy flow across r_h and the production of energy at the centre (see Lec 2).

The tidal radius

Indefinite expansion limited by the environment. Eventually, forces due to the Galaxy become important.

Notation:

M_G	mass of Galaxy (approximated as point mass)
R_G	galactocentric distance
M_C	cluster mass
r	distance of outlying cluster member from cluster centre (approximate cluster as point mass)

Force (per unit mass) due to galaxy dominates force due to cluster if

$$\frac{GM_G}{R_G^2} > \frac{GM_C}{r^2}$$

$$\Rightarrow r > R_G \left(\frac{M_C}{M_G}\right)^{1/2}$$

This is a **wrong** estimate of the size of a star cluster.

The tidal force

If the galactic force (per unit mass) is taken as $\dfrac{GM_G}{R_G^2}$, all stars in the cluster have the same galactic acceleration, and internal motion of the cluster is unaffected. We must estimate the *variation* of the galactic force between the position of the star and the position of the cluster centre, i.e.

$$r\frac{d}{dR_G}\left(\frac{GM_G}{R_G^2}\right) = -2r\frac{GM_G}{R_G^3}.$$

This differential force is called the *tidal force* (per unit mass). Hence effect of galaxy dominates if

$$\begin{aligned} \frac{rGM_G}{R_G^3} &\gtrsim \frac{GM_C}{r^2} \\ \Rightarrow r &> R_G\left(\frac{M_C}{M_G}\right)^{1/3} \end{aligned}$$

The tidal radius (corrected)

The limiting radius (beyond which acceleration due to Galaxy dominates) is the *tidal radius*, estimated by

$$r_t \simeq R_G \left(\frac{M_C}{M_G} \right)^{1/3} .$$

If a star cluster is born much smaller than its tidal radius ($r_h \ll r_t$) it will expand as in eq.(1) until r_h, r_t are comparable. At that point the cluster becomes *tidally limited*.
A tidally-limited cluster in a given galactic environment has constant density:

$$\frac{M_C}{r_t^3} \simeq \frac{M_G}{R_G^3}$$

Evolution of a tidally limited cluster

The cluster still tries to expand on a time scale t_{rh}, which causes escape across the tidal boundary:

$$\frac{\dot{M}}{M} \simeq -\frac{\xi}{t_{rh}},$$

where ξ is some constant.
Since $t_{rh} \propto M^{1/2} r_h^{3/2}$ and $M \propto r_h^3$ (constant density),

$$\frac{t_{rh}(t)}{t_{rh}(t_1)} \simeq \frac{M(t)}{M(t_1)},$$

where t_1 is some other reference time,

$$\frac{\dot{M}}{M(t)} \simeq -\frac{\xi}{t_{rh}(t_1)} \frac{M(t_1)}{M(t)}.$$

Evolution of a tidally limited cluster (continued)

$$\begin{aligned} \dot{M} &\simeq -\frac{\xi M(t_1)}{t_{rh}(t_1)} \\ \Rightarrow M(t) &\simeq M(t_1)\left(1-\xi\frac{t-t_1}{t_{rh}(t_1)}\right). \end{aligned}$$

The cluster "evaporates" at a finite time $t = t_1 + \dfrac{t_{rh}(t_1)}{\xi}$. Because density is constant, r_h also vanishes at that time; also t_{rh}:

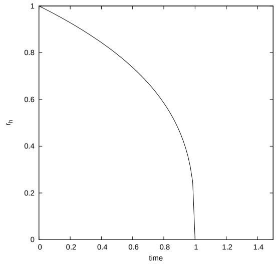
$$r_h(t) \simeq r_h(t_1)\left(1-\xi\frac{t-t_1}{t_{rh}(t_1)}\right)^{1/3}$$

Similarly $$t_{rh}(t) \simeq t_{rh}(t_1)\left(1-\xi\frac{t-t_1}{t_{rh}(t_1)}\right)$$

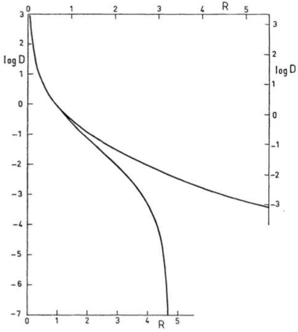
Hénon’s model

A numerical solution of the Fokker-Planck equation.

M. Henon, 1961, *Sur l’évolution dynamique des amas globulaires*, Annales d’Astrophysique, **24**, 369 (transl. by F. Renaud in arXiv:1103.3499).



(a) Evolution of r_h



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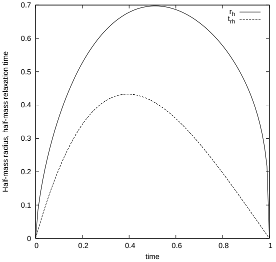
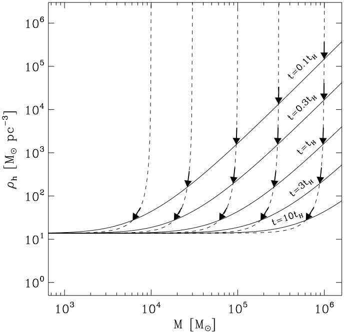
(b) Density profile (lower curve)

The entire life history

Combining these results gives an approximate picture of the entire evolution (Gieles et al, 2011, arXiv:1101.1821)



(a) Evolution of r_h, t_{rh}



(b) Evolutionary tracks and isochrones

A Monte Carlo model of M4

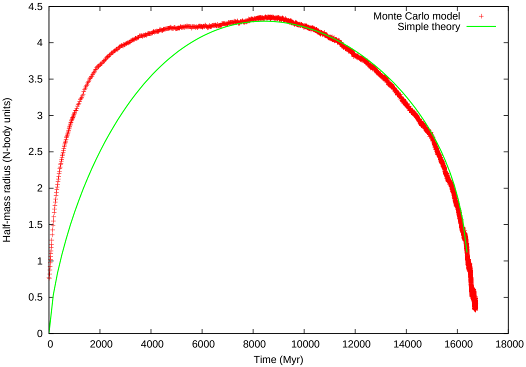


Figure: Evolution of the half-mass radius in a Monte Carlo model of M4 (Heggie & Giersz, 2008, MNRAS, 389, 1858), compared with simple theory.

The effect of the changing mass function, which is important in the early evolution, is ignored in the simplest version of the theory

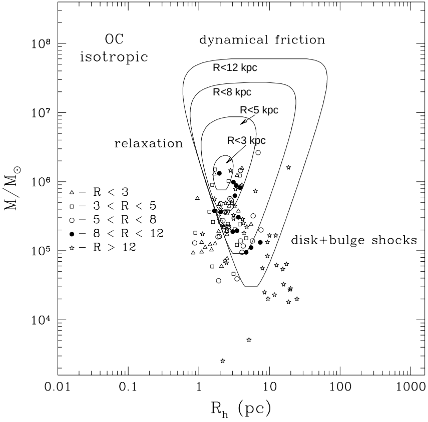
Application to the Milky Way system of globular clusters

Destruction mechanisms

- escape by dynamical interactions; time scale $\sim t_{rh}$
- unsteady tides ("shocks") when a cluster traverses the Galactic plane, or passes the bulge
- dynamical friction, which causes a massive cluster to spiral towards the Galactic Centre

Theoretical estimates of the time scales, expressed in terms of r_h and M, lead to a "survival triangle" inside which the time scales exceed a Hubble time (Fall & Rees, 1977, MNRAS, 181P, 37; Gnedin & Ostriker, 1997, ApJ, 474, 223).

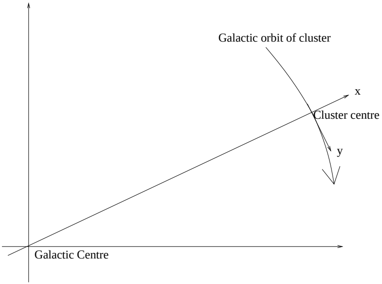
The survival triangle



In fact, clusters which are not tidally limited lie on the line $t_{rh} \simeq \frac{3}{2}\zeta t$ (eq.2)

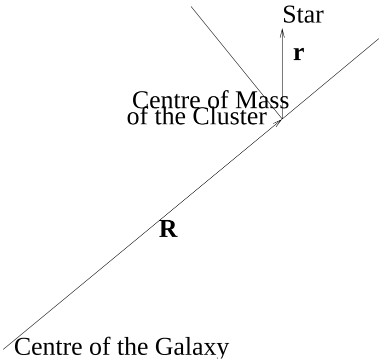
External Forces - the Galactic Tide

- Consider a cluster moving on a circular orbit about the Galactic Centre
- Use a rotating, accelerating frame of reference with origin at the centre of the cluster, and x-axis pointing to the Galactic Centre.



- This introduces centrifugal and Coriolis accelerations.

External Forces - the Galactic Tide (continued)



Position vector of the star is **R** + **r**, where

- **R** is the position of the centre of mass of the cluster with respect to the centre of the Galaxy.
- **r** is the position of a star relative to the centre of mass of the cluster

External Forces - the Galactic Tide (continued)

Position vector of the star is $\mathbf{R} + \mathbf{r}$

Velocity of the star is $\omega \times (\mathbf{R} + \mathbf{r}) + \dot{\mathbf{r}}$, where

- ω is the angular velocity of the cluster about the centre of the Galaxy
- $\dot{\mathbf{r}}$ is the vector whose components are the time derivatives of the components of $\mathbf{r}$ in the rotating frame

External Forces - the Galactic Tide (continued)

Position vector of the star is $\mathbf{R} + \mathbf{r}$

Velocity of the star is $\omega \times (\mathbf{R} + \mathbf{r}) + \dot{\mathbf{r}}$, where

- ω is the angular velocity of the cluster about the centre of the Galaxy
- $\dot{\mathbf{r}}$ is the vector whose components are the time derivatives of the components of $\mathbf{r}$ in the rotating frame

Similarly the acceleration is

$$
\begin{aligned}
& \omega \times (\omega \times (\mathbf{R} + \mathbf{r}) + \dot{\mathbf{r}}) + \frac{\partial}{\partial t}(\omega \times (\mathbf{R} + \mathbf{r}) + \dot{\mathbf{r}}) \\
= \quad & \omega \times (\omega \times (\mathbf{R} + \mathbf{r}) + \dot{\mathbf{r}}) + (\omega \times \dot{\mathbf{r}}) + \ddot{\mathbf{r}}) \\
= \quad & \omega \times (\omega \times (\mathbf{R} + \mathbf{r})) + 2\omega \times \dot{\mathbf{r}} + \ddot{\mathbf{r}} \\
= \quad & -\nabla \phi_C - \nabla \phi_G(\mathbf{R} + \mathbf{r})
\end{aligned}
$$

where

- ϕ_C is the gravitational potential due to the other stars in the cluster
- ϕ_G is the gravitational potential due to the Galaxy

External Forces - the Galactic Tide (continued)

Equation of motion of star:

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\mathbf{R}+\mathbf{r}))+2\boldsymbol{\omega}\times\dot{\mathbf{r}}+\ddot{\mathbf{r}}=-\nabla\phi_C-\nabla\phi_G(\mathbf{R}+\mathbf{r})$$

Equation of motion of (centre of mass of) cluster:

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})=-\nabla\phi_G(\mathbf{R})$$

Subtracting,

$$\begin{aligned}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+2\boldsymbol{\omega}\times\dot{\mathbf{r}}+\ddot{\mathbf{r}} &= -\nabla\phi_C-\nabla\phi_G(\mathbf{R}+\mathbf{r})+\nabla\phi_G(\mathbf{R})\\ &\simeq -\nabla\phi_C-\mathbf{r}.\nabla\nabla\phi_G(\mathbf{R})\end{aligned}$$

$$\Rightarrow \ddot{\mathbf{r}}+2\boldsymbol{\omega}\times\dot{\mathbf{r}}+[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\mathbf{r}.\nabla\nabla\phi_G(\mathbf{R})]=-\nabla\phi_C$$

External Forces - the Galactic Tide (continued)

Equation of motion of star:

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{R} + \mathbf{r})) + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \ddot{\mathbf{r}} = -\nabla\phi_C - \nabla\phi_G(\mathbf{R} + \mathbf{r})$

Equation of motion of (centre of mass of) cluster:

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) = -\nabla\phi_G(\mathbf{R})$

Subtracting,

$$
\begin{aligned}
\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \ddot{\mathbf{r}} &= -\nabla\phi_C - \nabla\phi_G(\mathbf{R} + \mathbf{r}) + \nabla\phi_G(\mathbf{R}) \\
&\simeq -\nabla\phi_C - \mathbf{r}.\nabla\nabla\phi_G(\mathbf{R})
\end{aligned}
$$

$$
\Rightarrow \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r}.\nabla\nabla\phi_G(\mathbf{R})] = -\nabla\phi_C
$$

Interpretation:

$2\boldsymbol{\omega} \times \dot{\mathbf{r}}$	Coriolis acceleration
$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$	centrifugal acceleration
$\mathbf{r}.\nabla\nabla\phi_G(\mathbf{R})$	Galactic tidal acceleration

External Forces - the Galactic Tide (continued)

Centrifugal and tidal accelerations derivable from a potential:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r}.\nabla\nabla\phi_G(\mathbf{R}) = \nabla \left\{ \frac{1}{2}(\boldsymbol{\omega}.\mathbf{r})^2 - \frac{1}{2}\omega^2 r^2 + \frac{1}{2} x_i \frac{\partial^2 \phi_G}{\partial x_i \partial x_j} x_j \right\},$$

where $\mathbf{r} = (x_1, x_2, x_3)$, $r = |\mathbf{r}|$.

If ϕ_G is the potential of an axisymmetric galaxy with a plane of symmetry, the tidal potential reduces to two terms.

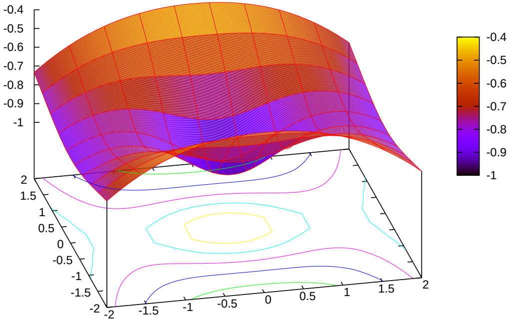
Example: point-mass galaxy:

$$\begin{aligned}
\ddot{x} + 2\omega\dot{y} - 3\omega^2 x &= -\frac{\partial \phi_C}{\partial x} \\
\ddot{y} - 2\omega\dot{x} &= -\frac{\partial \phi_C}{\partial y} \\
\ddot{z} + \omega^2 z &= -\frac{\partial \phi_C}{\partial z}
\end{aligned}$$

For isothermal galaxy model $-3\omega^2 x \to -2\omega^2 x$, etc.

External Forces - the Galactic Tide (continued)

- Potential and equipotentials in the plane of motion of the cluster



- note equilibria (Lagrange points) which mark the tidal radius. Potential here defines escape energy E_e.

Tidal effects

Neglect acceleration due to cluster (e.g. distant star or escaper):

$$\begin{aligned} \ddot{x} + 2\omega\dot{y} - 3\omega^2 x &= 0 \\ \ddot{y} - 2\omega\dot{x} &= 0 \end{aligned}$$

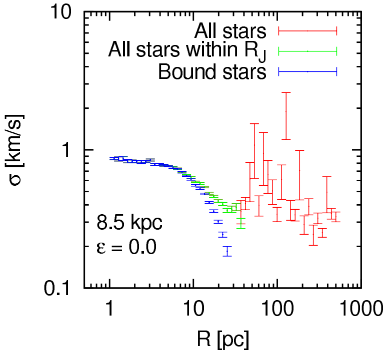
if $z = 0$ (star moving in orbital plane of cluster). Motion on an ellipse is possible:

$$\begin{aligned} x &= a\cos\omega t \\ y &= 2a\sin\omega t \end{aligned}$$

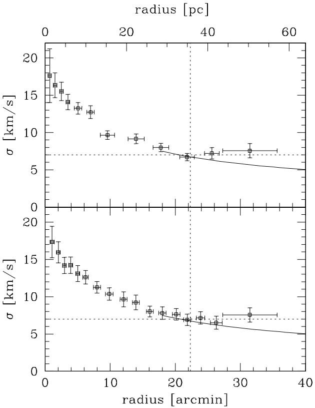
In fact this family of motions (including the cluster field) persists inside the tidal radius, and have energy above the escape energy E_e. In simulations, a few percent of stars inside the tidal radius have energies $> E_e$ (“potential escapers”).

Effect of Potential Escapers

Velocity dispersion profile in *N*-body model and some observed clusters



(a) *N*-body model (Kuepper et al, 2010, MNRAS, 407, 2241)



(b) ω Cen (Scarpa & Falomo, 2010, A&A, 523A, 43

Tidal tails

Equations of motion far from the cluster:

$$
\begin{aligned}
\ddot{x} + 2\omega\dot{y} - 3\omega^2 x &= 0 \\
\ddot{y} - 2\omega\dot{x} &= 0
\end{aligned}
$$

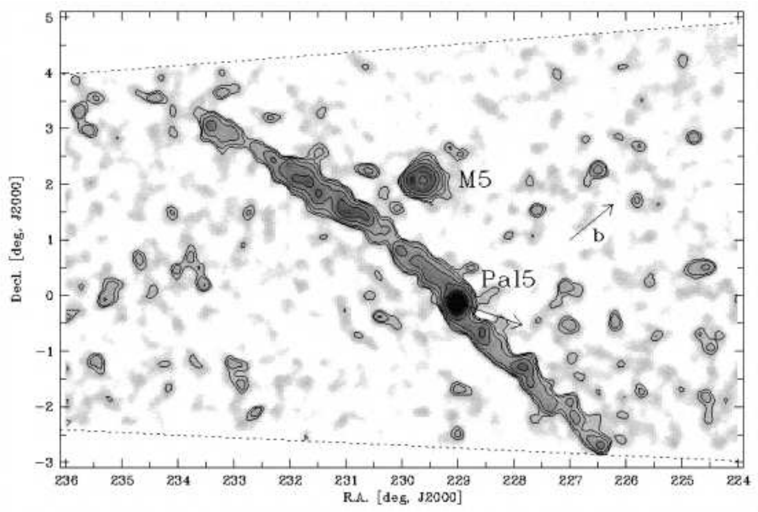
have solutions

$$
\begin{aligned}
x &= c \quad \text{(constant)} \\
y &= \frac{3}{2}c\omega t.
\end{aligned}
$$

These straight-line motions are followed by escapers in *N*-body simulations. (Movie - starlab)

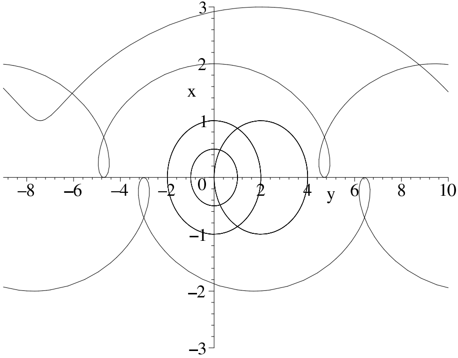
Tidal tails

The example of Pal 5 (Source: Odenkirchen et al, AJ, 126.2385)



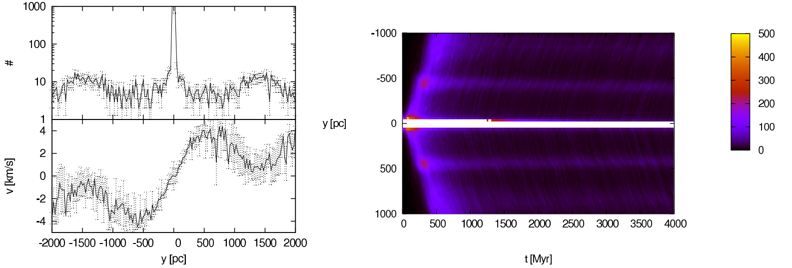
The structure of tidal tails

The general solution of the equations in a tidal field (neglecting cluster acceleration) is superposition of elliptic and straight-line motion, called “epicyclic motion”.



Speed decreases near “loops”, particles accummulate there, leading to density enhancements.

The structure of tidal tails (simulations)

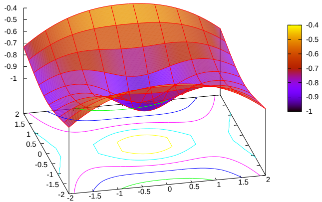


(c) (d)

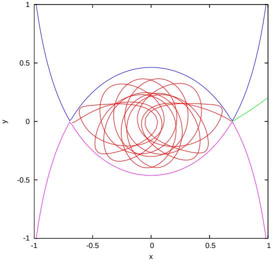
Kuepper et al, 2010, MNRAS, 401, 105

Effect of potential escapers on the escape time scale

Stars inside the tidal radius with $E > E_c$ (potential escapers) may remain inside r_t indefinitely, or at least for a long time:



(e) The potential well



(f) An orbit of a potential escaper

Effect of potential escapers on the escape time scale (cont)

Time it takes a potential escaper of energy $E > E_e$ to escape is
$t_E \propto (E - E_e)^{-2}$
(Fukushige & Heggie 2000, MNRAS, 318, 753).
In this time, relaxation changes the energy of a star by
$\Delta E \propto \left(\frac{t_E}{t_{rh}}\right)^{1/2}$ (since relaxation is diffusive).
Set $E - E_e = \Delta E$. Thus $t_E^{-1/2} \propto t_E^{1/2} t_{rh}^{-1/2}, \Rightarrow t_E \propto t_{rh}^{1/2}$. Hence
$\Delta E \propto t_{rh}^{-1/4}$.
The fraction of mass escaping in time t_E is proportional to ΔE, and so the time scale of escape is proportional to

$$\frac{t_E}{\Delta E} \propto \frac{t_{rh}^{1/2}}{t_{rh}^{-1/4}} = t_{rh}^{3/4}$$

(Baumgardt, 2001, MNRAS, 325, 1323)

Summary of Lecture 1

- The half-mass radius r_h is the radius of a sphere containing half the mass (theoretical equivalent of observer's half-light radius)
- r_h evolves on the half-mass relaxation time scale
- For a compact cluster (tide can be neglected) r_h expands as $t^{2/3}$
- For a tidally limited cluster, $M/r_h^3 \simeq$ constant; r_h varies as $(t_{diss} - t)^{1/3}$, where t_{diss} is time of dissolution
- *For a tidally limited cluster, dissolution time* $\propto t_{rh}^{3/4}$
- *Stars escape along tidal tails (which follow the orbit of the cluster through the Galaxy)*
- *Tidal tails may be clumpy through purely gravitational effects.*

Lecture 2: Fundamental aspects of core radius evolution

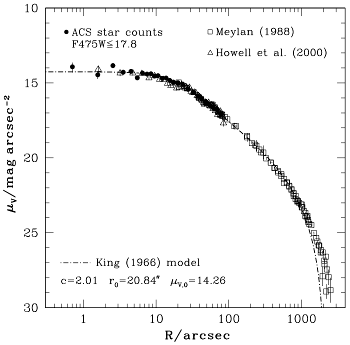


Figure: Surface brightness profile of 47 Tuc (McLaughlin et al, 2006)

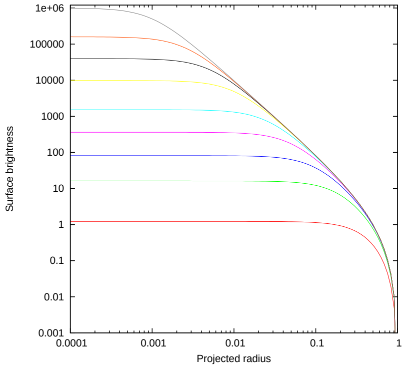
- Core radius $r_c = 22''$
- Half-mass radius $r_h = 190''$ (half-light radius)
- Tidal radius $r_t = 2500''$

The core radius

- Traditional observer's definition: radius where surface brightness falls to half its central value.
- Theorist's value:
$$r_c = \left(\frac{9\sigma_c^2}{4\pi G\rho_c}\right)^{1/2}$$
(King, 1966, AJ, 71, 64)[2], where
  - σ_c is the rms random velocity in one coordinate, at the centre
  - ρ_c is the central mass density
- note some ambiguities:
  - ρ may be dominated by dim stars (white dwarfs, neutron stars), and so the observer's core radius may be substantially different
  - is σ mass-weighted?
- note: three-dimensional velocity dispersion is $3\sigma^2$ if the distribution of velocities is isotropic

[2] 1463 citations on 30/4/11

What determines the core radius?



King templates with the same r_t, roughly similar r_h, and widely different r_c (King, 1962, AJ, 67, 471) [3]

[3] Only 1065 citations on 30/4/11

Energy generation by stellar evolution

Stars lose considerable mass in late stages of evolution, e.g. in supernovae.

Suppose a star loses mass $\delta m > 0$, which escapes rapidly and isotropically. Then change in energy of the cluster is

$$\delta E = -\frac{1}{2}(\delta m)v^2 + G\sum_i{}' \frac{(\delta m)m_i}{r_i},$$

where

- v is the speed of the star
- $\sum'$ is a sum over all other stars i
- m_i, r_i are mass and distance to ith star

First term generally considerably smaller, hence

$$\delta E \simeq -(\delta m)\phi$$

where $\phi < 0$ is potential (per unit mass)

Energy generation by stellar evolution (cont)

Rate of change of energy is

$$\dot{E} = \dot{M}\langle\phi\rangle,$$

where $\langle\phi\rangle$ is the mean potential of the stars which are losing mass. These are the most massive unevolved stars, which are concentrated near the centre, because of mass segregation. $|\phi|$ largest at centre, and value depends on size of core.
Energy is flowing across the half-mass radius at a rate of order $\zeta|E|/t_{rh}$ (slide 2).

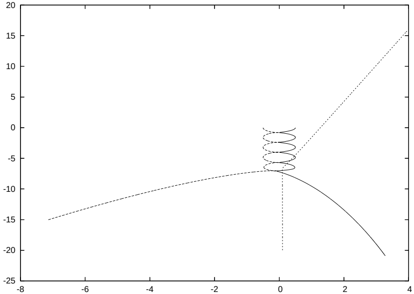
Hénon’s Principle (Hénon, 1975, IAUS, 69, 133)
..the rate of flow of the energy is controlled by the system as a whole, not by the singularity...

Application of Hénon's Principle to stellar evolution

- The system as a whole requires energy flow $\dot{E} \simeq \zeta \dfrac{|E|}{t_{rh}}$.
- The core produces energy at a rate $\dot{M}\langle\phi\rangle$.
- If the core is producing too much energy, it adjusts so as to reduce $\dot{E}$, i.e. $|\langle\phi\rangle|$ reduces, i.e. the core radius expands.
- If the core is producing too little energy, then the core is losing energy, and the core radius decreases.
- If the core can adjust to make $\dot{M}\langle\phi\rangle \simeq \zeta \dfrac{|E|}{t_{rh}}$, we have "balanced evolution" (Lecture 1).

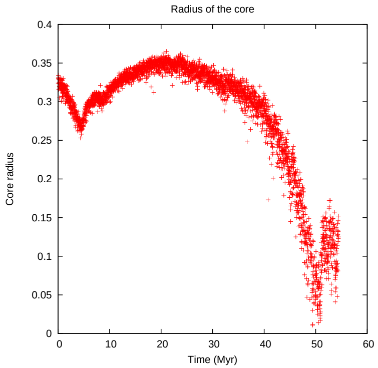
Binary formation

- If there is no energy generating mechanism, the core will contract (core collapse)
- Eventually the density becomes so high that 3-body interactions become significant. Then a binary will form. This process is exothermic, and provides the required energy.



Two movies illustrating core collapse (no energy generation)

An illustrative *N*-body model



- No mass loss up to 5Myr – core collapse
- Mass loss after 5Myr, but efficiency decreases; initial core expansion, then recollapse. Most massive component is black holes, which mass-segregate, but lose no further mass.
- black holes core-collapse, create BH binaries, and reverse cc

Energy generation by hypothetical IMBH

- IMBH – intermediate-mass black hole (a few thousand $M_\odot$; see talk by Marzieh Bahmani)
- for high probabity of capture by the IMBH (coallescence), a star must be strongly bound to it (large negative energy). Stars reach large negative energy by relaxation, and the scattered stars gain energy.
- Near the black hole $\sigma^2 \sim GM_\bullet/r$
- Energy of matter near radius r is $\sim \rho r^3\sigma^2$
- Relaxation time $\sim \sigma^3/(G^2 m\rho \ln\Lambda)$
- Energy flux $\dot{E} \sim \rho^2 r^3 G^2 m \ln\Lambda/\sigma \sim \rho^2 r^{7/2} G^2 m \ln\Lambda/\sqrt{GM_\bullet}$
- For constant flux $\rho \propto r^{-7/4}$ (Bahcall & Wolf , 1976, ApJ, 209,214) – a cuspy density distribution
- Evaluate at the edge of the cusp, where ρ, σ approach values in the core: $\dot{E} \sim \rho_c^2 G^5 M_\bullet^3 m \ln\Lambda/\sigma^7$
- For given flux (Hénon's Principle) large BH mass requires low ρ_c, i.e. large core radius (Heggie et al, 2007, PASJ, 59L, 11)

Dynamical ejection of binary stars

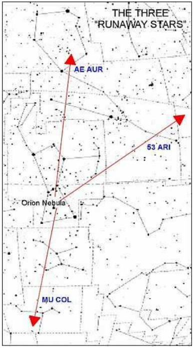


Figure: Runaway stars from the Orion star forming region (http://www.weasner.com/etx/ref_guides/auriga.html)

This can be interpreted as a result of a dynamical binary-binary interaction (Gualandris et al, MNRAS, 2004, 350, 615). The loss of mass from a cluster is another heating mechanism.

Escape by energetic three-body interactions

- Assume all masses equal. Binary star with semi-major axis a has binding energy $\dfrac{Gm^2}{2a}$. Typical orbital speed is of order $\sqrt{Gm/a}$. Suppose $\sqrt{Gm/a} \gg \sigma$ (rms 1-dimensional speed of single stars in a cluster); i.e. a *hard* binary
- If binary and a single star, initially moving with speed $\sim \sigma$, interact within a distance $\sim a$, star will typically leave with a speed $\sim \sqrt{Gm/a} \gg \sigma$.
- By conservation of momentum, binary must also leave with comparable high speed.
- We suppose a small enough that the three objects escape from the cluster, and estimate the cross section

Estimate of reaction and heating rates

- Cross section $\Sigma \sim \pi a^2\left(1 + \dfrac{6Gm}{aV_\infty^2}\right)$ (lectures by M. Davies)
- Since we assumed $Gm/a \gg \sigma^2 \sim V_\infty^2$, cross section much greater than "size" of binary (πa^2), and $\Sigma \simeq 6\pi Gma/V_\infty^2$.
- The binary sweeps out a cylinder of cross-sectional area Σ at speed V_∞. Hence number of close encounters per unit time is

$$\dot{N}_{enc} \simeq n\Sigma V_\infty,$$

  where n is the number density of single stars.
- If all three stars escape after the encounter, heating rate is

$$\begin{aligned} \dot{E} &\simeq 3m|\phi|n\Sigma V_\infty \\ &\simeq 3m|\phi|n\frac{6\pi Gma}{V_\infty^2}V_\infty \\ &\sim -\frac{Gm^2an\phi}{\sigma} \end{aligned}$$

Binary heating

For a single binary

$$\dot{E} \sim -Gm^2 n a \phi / \sigma \tag{3}$$

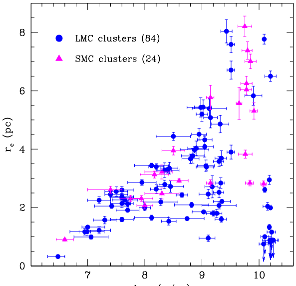
- Heating also occurs in encounters even if there is no escape. Changes only the numerical coefficient. (movie)
- Suppose there are many binaries, with binary fraction f. (observations imply $f \lesssim 0.1$; see slide 2). Total number of binaries in the core is $\sim f n r_c^3$, and energy generation rate is

$$\begin{aligned}
\dot{E} &\sim f G m^2 n^2 a |\phi| r_c^3 / \sigma \\
&\sim \frac{f G m^2 n^2 a |\phi|}{\sigma} \frac{\sigma^3}{(Gnm)^{3/2}} \quad \text{by eq.(2)} \\
&\sim f a |\phi| \sigma^2 \sqrt{mn/G}
\end{aligned}$$

Again $\dot{E}$ increases if the core contracts (as n and $|\phi|$ increase)

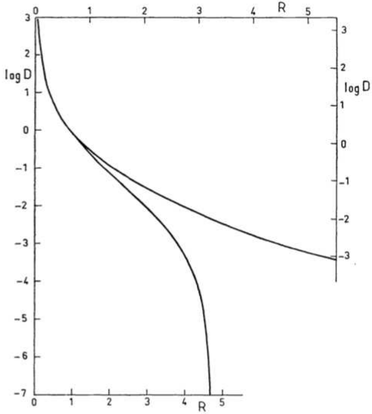
Core evolution by mass segregation

- As massive stars segregate to the centre, they lose energy.
- This energy is gained by stars of lower mass.
- If the massive stars have low luminosity (e.g. black holes), the visible core of luminous stars expands. (Merritt et al, ApJ, 2004, 608L, 25; Mackey et al, 2008, MNRAS, 386, 65)
- May relate to trend of increasing core radius with age in star clusters of the Magellanic Clouds
- Formation and evolution of black hole binaries also important



The stability of balanced evolution: The spatial structure

- Balanced evolution: the core is producing energy at the rate determined by conditions at (say) the half-mass radius.
- Spatial structure is like Hénon's models (Lecture 1), except for the presence of a core:



Isothermal models

- Except at large r, Hénon's model is nearly isothermal (σ constant, i.e. independent of r)
- Hydrostatic equation (also one of Jeans' equations of stellar dynamics)

$$\frac{dp}{dr} = -\frac{d\phi}{dr}\rho$$

where

- $p = \rho\sigma^2$ is the (kinetic) pressure
- ϕ is the gravitational potential per unit mass

Hence

$$\begin{aligned} \frac{\sigma^2}{\rho}\frac{d\rho}{dr} &= -\frac{d\phi}{dr} \\ \Rightarrow \sigma^2 \ln\left(\frac{\rho}{\rho_0}\right) &= -\phi \end{aligned}$$

where ρ_0 is the central density, and we assume $\phi = 0$ there.

The isothermal equation

$$\sigma^2 \ln\left(\frac{\rho}{\rho_0}\right) = -\phi$$
$$\Rightarrow \rho = \rho_0 \exp\left(-\frac{\phi}{\sigma^2}\right)$$

Potential gradient given by “Newton’s Second Theorem”

$$\frac{d\phi}{dr} = \frac{GM(r)}{r^2} \quad (M(r) \text{ is mass within radius } r)$$
$$\Rightarrow M(r) = \frac{r^2}{G}\frac{d\phi}{dr}$$
$$\Rightarrow 4\pi r^2\rho = \frac{1}{G}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \quad (\text{differentiating with respect to } r)$$
$$\Rightarrow 4\pi G r^2\rho = 2r\frac{d\phi}{dr} + r^2\frac{d^2\phi}{dr^2} \quad (\text{a form of Poisson's equation})$$

The isothermal equation (continued)

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = 4\pi G\rho$$
$$= 4\pi G\rho_0 \exp\left(-\frac{\phi}{\sigma^2}\right)$$

Scale $\phi = y\sigma^2$. Thus

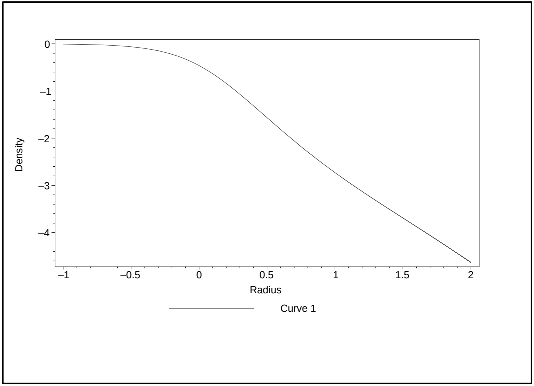
$$\frac{d^2y}{dr^2} + \frac{2}{r}\frac{dy}{dr} = \frac{4\pi G\rho_0}{\sigma^2}\exp(-y)$$

Scale $r = r_c x = \sqrt{\frac{9\sigma^2}{4\pi G\rho}}$ (eq.2). Then

$$\boxed{\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 9\exp(-y).}$$ (the *isothermal equation*, except for the 9)

The isothermal solution

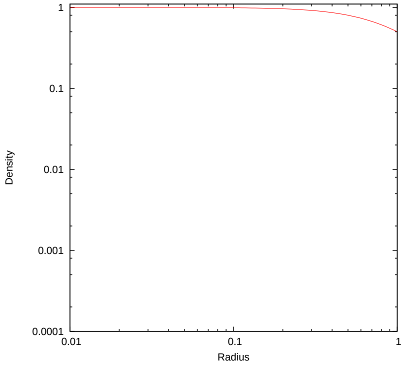
Log-log plot of density against radius



At large radii $\rho \propto r^{-2}$, $M(r) \propto r$, infinite total mass.

Stability of an isothermal model

- Isothermal model has infinite mass and radial extent, and so we consider a *truncated* model, enclosed within a rigid, spherical, adiabatic boundary. No energy or mass can escape.
- Suppose the boundary radius is small; say, comparable with r_c



Stability of an isothermal model (continued)

- When the boundary radius is small, the density contrast between centre and boundary is small.
- The effect of gravity is weak
- The system behaves like an ordinary gas (though weakly gravitating) and is stable.

Stability of an isothermal model (continued)

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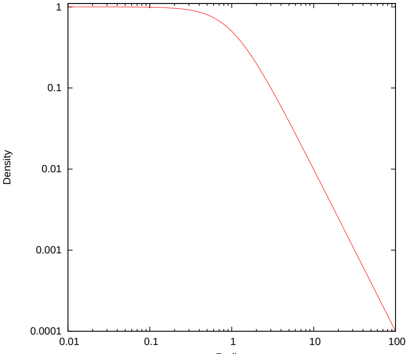
Systems with large density contrast



Stability of an isothermal model (continued)

- imagine some heat shifted from the inside to the outside.
- The outside is kept in by the boundary. The temperature rises, but only slighty, because the outer parts of the model are so massive (high heat capacity)
- The inside is self-gravitating. It loses heat, particles fall closer to the centre, and gain more kinetic energy than they lost (negative heat capacity of a self-gravitating system)[4]
- The centre is now hotter than the outer parts, and heat continues to flow from centre to outer parts.
- A *gravothermal runaway* is created, i.e. the model is unstable.
- Core collapse can be interpreted as a result of gravothermal runaway

---

[4] In a self-gravitating system in virial equilibrium, $T = -E$, where T is kinetic energy and E is total energy. If E decreases, T increases.

Stability of an isothermal model (continued)

- An isothermal model with a density contrast $\dfrac{\rho(0)}{\rho(\mathrm{edge})} \gtrsim 709$ is gravothermally unstable (Antonov, 1962; transl. in 1975, IAUS, 113, 525)
- Physical explanation due to Lynden-Bell & Wood, 1968, MNRAS, 138, 495
- For a fascinating, readable account of the thermodynamics of self-gravitating systems, see Padmanabhan, 1990, Phys. Rep., 188, 285

Stability of balanced evolution - example

- Consider a single binary at the centre of a star cluster
- $\dot{E} \sim -\dfrac{Gm\rho_c a\phi_c}{\sigma}$ (eq.3)
- This must balance energy flow at half-mass radius, i.e.

$$-\frac{Gm\rho_c a\phi_c}{\sigma} \sim -\frac{E}{t_{rh}}.$$

- We estimate

$$\begin{aligned} E &\sim -\frac{GM^2}{r_h} \\ t_{rh} &\sim \frac{\sigma^3}{G^2 m\rho_h \ln\Lambda} \\ \Rightarrow -\frac{Gm\rho_c a\phi_c}{\sigma} &\sim \frac{G^3M^2 m\rho_h \ln\Lambda}{r_h\sigma^3} \end{aligned}$$

Stability of balanced evolution - example (continued)

$$
\begin{aligned}
-\frac{Gm\rho_c a\phi_c}{\sigma} &\sim \frac{G^3M^2m\rho_h\ln\Lambda}{r_h\sigma^3} \\
\Rightarrow \frac{\rho_c}{\rho_h} &\sim -\frac{G^2M^2\ln\Lambda}{r_h\sigma^2 a\phi_c} \\
&\sim \left(\frac{\sigma^2}{-\phi_c}\right)\left(\frac{Gm}{a\sigma^2}\right)\left(\frac{GM\ln\Lambda}{r_h\sigma^2}\right)N
\end{aligned}
$$

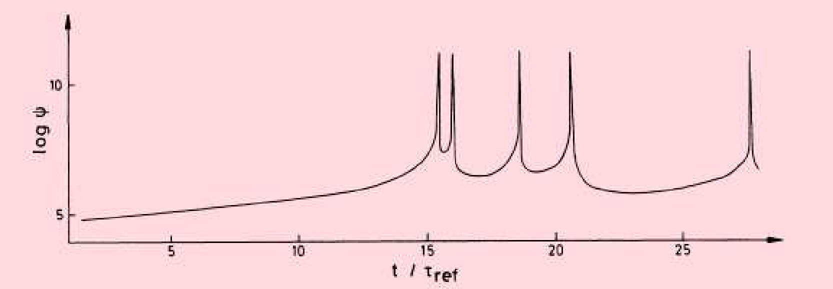
where

- ϕ varies logarithmically (i.e. weakly) in an isothermal region, so first factor almost constant
- Second factor is hardness of the binary, which we treat as constant
- Third factor is almost constant (virial theorem)

Therefore we expect balanced evolution to be unstable if N is sufficiently large (as this makes the density contrast large)

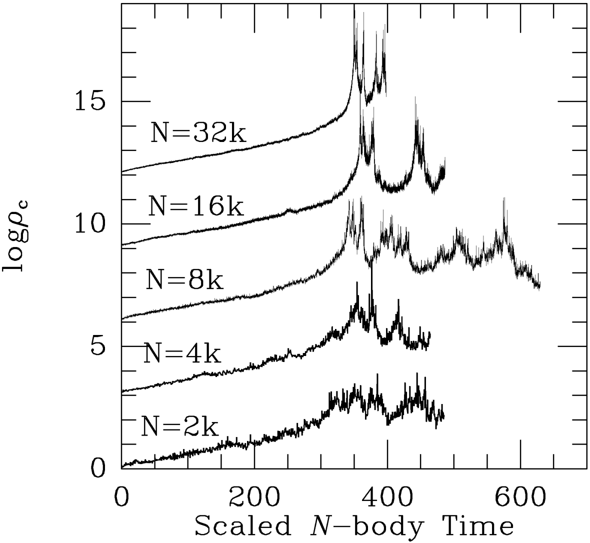
Gravothermal oscillations

- For binaries formed in 3-body encounters, and equal masses, the critical $N \sim 7000$ (Goodman, 1987, ApJ, 313, 576)
- The instability was discovered by Sugimoto & Bettwieser, 1983, MNRAS, 204P, 19; see also talk by Phil Breen



Dependence of gravothermal oscillations on *N*

N-body models: Makino, 1996, ApJ, 471, 796



Gravothermal oscillations in a model of NGC6397

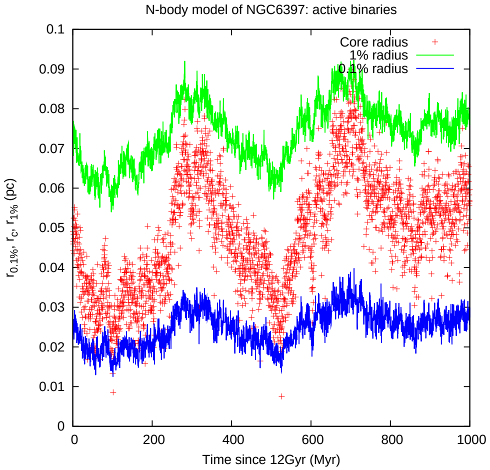


Figure: Heggie & Giersz, 2009, MNRAS, 397L, 46

Summary of Lecture 2

- Definition of the core radius
- Energy generation mechanisms: mass loss by stellar evolution, action of a central IMBH, ejection of binaries, hardening of binaries; evolution of core by mass segregation of a dark component; formation of binaries
- Hénon's Principle
- Binary interactions
  - gravitational focusing
  - close-interaction cross section
  - *hard and soft binaries and their average behaviour*
- Isothermal models: stability
- Core radius in balanced evolution
- Gravothermal oscillations

Lecture 3: Modelling globular clusters and their long-term evolution

Two types:

1. "Snapshot models": dynamically consistent models which describe the distribution of stars, in phase space, at the present day. Examples:
   - King's models (King 1966) and their variants
   - Schwarzschild models (e.g. van de Ven et al, 2006, A&A, 445, 513 – applied to ωCen)
   - Non-parametric modelling with Jeans' equations (e.g. Gebhardt & Fischer, AJ, 109, 209)
2. Evolutionary models: models which follow the dynamical and stellar evolution from some initial state to the present day. Examples:
   - *N*-body models (Hasani Zonoozi et al, 2011, MNRAS, 411, 1989 – Pal 14)
   - Monte Carlo models (e.g. Giersz & Heggie, 2011, MNRAS, 410, 2698 – 47 Tuc)

Snapshot models
Example: Isothermal model

For isothermal,

$$\frac{\rho}{\rho_0} = e^{-y} \quad \text{(slide 1)}$$

where
$$y = \frac{\phi}{\sigma^2}$$

and
$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 9\frac{\rho}{\rho_0}$$

where
$$x = \frac{r}{r_c}$$

and so
$$\rho = \rho_0\, F\left(\frac{r}{r_c}\right),$$

where F is a certain function which can easily be calculated numerically.

Isothermal and King's models

Then the surface density, on a line of sight passing at a distance R from the centre, is

$$\begin{aligned}\Sigma &= \int_{-\infty}^{\infty} \rho\left(\sqrt{R^2 + z^2}\right) dz \\ &= \Sigma_0 G\left(\frac{R}{r_c}\right),\end{aligned}$$

where Σ_0 is the value at the centre, and G is another function.

Isothermal and King's models

Then the surface density, on a line of sight passing at a distance R from the centre, is

$$\begin{aligned} \Sigma &= \int_{-\infty}^{\infty} \rho\left(\sqrt{R^2 + z^2}\right) dz \\ &= \Sigma_0 G\left(\frac{R}{r_c}\right), \end{aligned}$$

where Σ_0 is the value at the centre, and G is another function.
For King's model (see talk by Marzieh Bahmani)

$$\frac{\rho}{\rho_0} = e^{-y} \frac{-\dfrac{\exp(-W_0 + y)}{3\sqrt{\pi}}\left(4(W_0 - y)^{3/2} + 6(W_0 - y)^{1/2}\right) + \operatorname{erf}\sqrt{W_0 - y}}{-\dfrac{\exp(-W_0)}{3\sqrt{\pi}}\left(4W_0^{3/2} + 6W_0^{1/2}\right) + \operatorname{erf}\sqrt{W_0}}$$

where W_0 is a parameter related to the depth of the potential well.

King's model (continued)

Then

$$\Sigma = \Sigma_0 \; F\left(\frac{R}{r_c}, W_0\right),$$

where F is a different function.

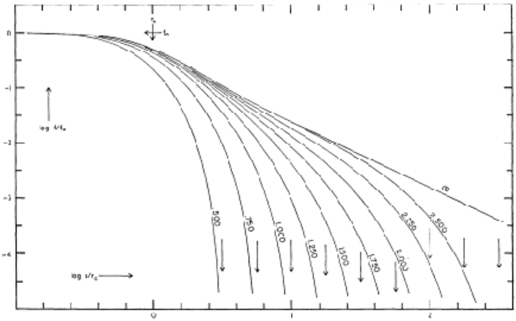
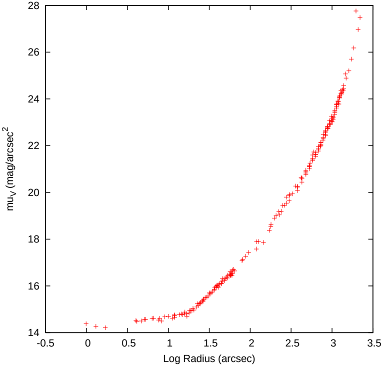


Figure: King (1966) models: fixed r_c, Σ_0, varying W_0

Model fitting

Source: Data from Trager et al, 1995, AJ, 109, 218 at http://vizier.u-strasbg.fr/viz-bin/VizieR?-source=J/AJ/109/218

Name	log Radius (arcsec)	μ_V (mag/arcsec^2)	Weight
ngc104	-0.005	14.38	1.000
ngc104	0.115	14.27	1.000
...	...	...	... (206 entries)



Model fitting (continued)

- Unit conversion required
  - parsec → arcsec (requires distance)
  - surface density → surface brightness (requires M/L_V ratio, and $M_{V\odot}$)
- require numerical measure of goodness of fit, e.g.

$$\Delta = \sum_{1}^{206} w_i(\mu_i - \mu(R_i, r_c, \rho_0, W_0))^2$$

- require optimisation routine for minimising Δ in the space of the three parameters r_c, ρ_c, W_0, e.g. amoeba (Press et al, 1992, Numerical Recipes, CUP)
- A more informative method of optimisation is MCMC (Markov Chain Monte Carlo). See for example CosmoMC (http://cosmologist.info/cosmomc/). Provides information on the range of parameter values consistent with the data, and correlations among them (see also Tom Kitching's Lecture 3).

Dynamical evolutionary models I: *N*-body methods

One Milky Way globular cluster has been modelled with full-size *N*-body model: Pal 14 (Hasani Zonoozi et al, 2011, MNRAS, 411, 1989)

The N-body problem

- Summary: integrate numerically the equations

$$\ddot{\mathbf{r}}_i = -G \sum_{j\neq i, j=1}^{j=N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

- Each force takes $\sim 30N$ calculations
- Each of N stars needs updating once per time step
- Typical time step $\ll$ time to close encounter with nearest neighbour, $\dfrac{r_h}{N^{1/3}\sigma} \sim \dfrac{t_{cr}}{N^{1/3}}$, where t_{cr} is the crossing time.
- About $N/\ln N$ crossing times per relaxation time ($t_{rh} \propto \dfrac{N^{1/2} r_h^{3/2}}{\ln N}$)
- Computing time (wall-clock time) $W \propto N^{10/3}/\ln N \sim N^3$ per relaxation time $\sim N^{5/2} r_h^{-3/2}$
- Binaries very time-consuming (factor $\gtrsim 10$ for few % binary fraction); dependence on N, r_h complicated by distribution of hardness

History of progess with the *N*-body problem

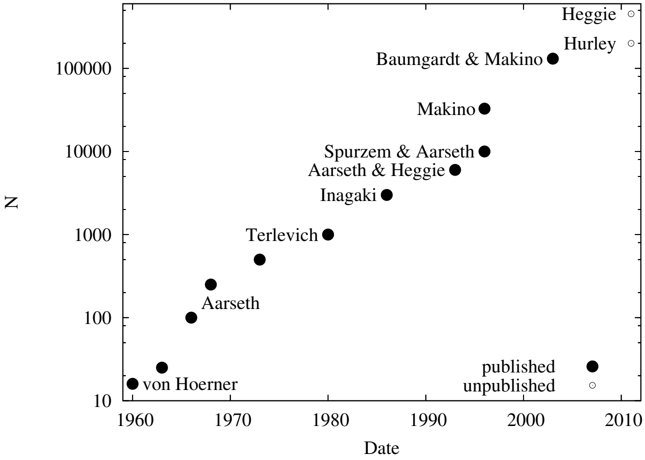
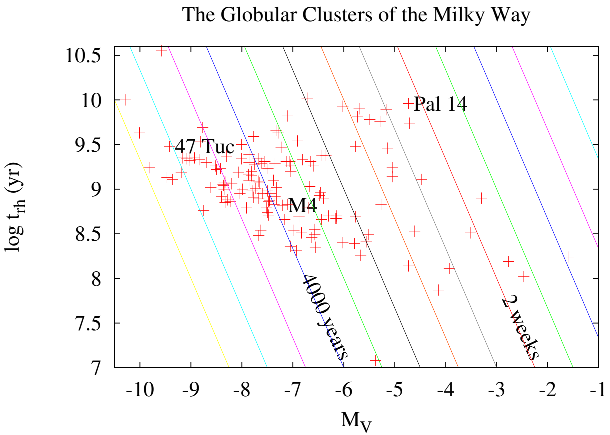


Figure: Largest *N*-body simulation to date showing advanced evolution (mostly published)

Estimates for *N*-body modelling of globular clusters

- Assume $W \propto N^{5/2} r_h^{-3/2}$
- Hasani Zonoozi et al found $W \simeq 1$ month for Pal 14 (with binaries; $r_h \simeq 34$pc, $M \sim 10^4 M_\odot$)



Small-scale *N*-body models

- The idea: model a cluster which has N stars by a model with $N^* \ll N$ stars
- The principle: get the time scale of the major evolutionary effects correct
  - Stellar evolution: set by stellar evolution models
  - Two-body relaxation: time scale $t_{rh} \propto \dfrac{N^{1/2} r_h^{3/2}}{\log N}$

    Hence choose radius of "small-scale" model so that

$$
\begin{aligned}
\frac{N^{*1/2} r_h^{*3/2}}{\log \gamma N} &= \frac{N^{1/2} r_h^{3/2}}{\log \gamma N} \\
\Rightarrow r_h^* &= r_h \left(\frac{N}{N^*}\right)^{1/3} \left(\frac{\log \gamma N^*}{\log \gamma N}\right)^{2/3}
\end{aligned}
$$

(Portegies Zwart et al, 1999, A&A, 348, 117; Heggie & Giersz, 2008, MNRAS, 389, 1858)

Dynamical evolutionary models II: Monte Carlo codes

- Aims to follow the evolution on time scale of $t_{rh} (\gg t_{cr})$
- Each star characterised by its radius r (distance from cluster centre), energy E and angular momentum J
- In each time step ($\propto t_{rh}$), for each star,
  - r chosen appropriately for an orbit of energy E and angular momentum J
  - E and J are changed slightly according to theory of relaxation (cumulative effect of gravitational encounters)
- Includes approximate treatments of
  - Binaries (interactions treated with cross sections or on-the-fly computation of individual interactions)
  - stellar evolution (as in N-body codes)
  - steady tide

Dynamical evolutionary models II: Monte Carlo codes (continued)

- Exclusions
  - unsteady tides
  - cluster rotation
  - non-spherical cluster
- Advantage: speed. Even the cluster 47 Tuc ($N \simeq 2 \times 10^6$) takes a few days (single core)
- Availability: soon available within AMUSE (see http://amusecode.org/)
- See
  - Giersz et al, 2008, MNRAS, 388, 429
  - Hénon, 1971, ApSS, 14, 151
  - CMC (Cluster Monte Carlo code, J. Fregeau, ex-Northwestern University)

Complexity of the Monte Carlo method

In each step

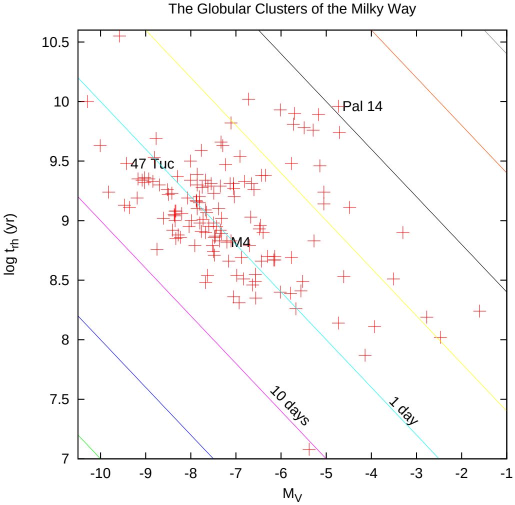
- Reassign new positions (radii) to all N stars ($O(N \log N)$)
- Order the N stars by radius (for rapid calculation of the potential and selection of particles for interaction) ($O(N \log N)$)
- Work through all $N/2$ pairs of neighbouring stars to carry out
  - Two-body relaxation
  - Stellar evolution
  - (If one of the "stars" is a binary) carry out binary-single interaction
  - (If both of the "stars" are binaries) carry out binary-binary interaction

Hence effort is

$$\begin{aligned} W &\sim N \log N \quad \text{per relaxation time } t_{rh} \propto N^{1/2} r_h^{3/2} \\ &\propto N^{1/2} r_h^{-3/2} \quad \text{(neglecting log factors)} \end{aligned}$$

Monte Carlo - time taken to simulate MW globular clusters

Assume wall clock time $\propto N^{1/2} r_h^{-3/2}$ (slide 3)



A Case Study: the Globular Cluster M4

Present-day parameters (Heggie & Giersz, 2008, MNRAS, 389, 1858)

- Distance from the sun 1.72 kpc (Harris catalogue)
- $N \simeq 80000$, $r_h \simeq 2.9$pc, $t_{rh} \simeq 3 \times 10^9$ yr

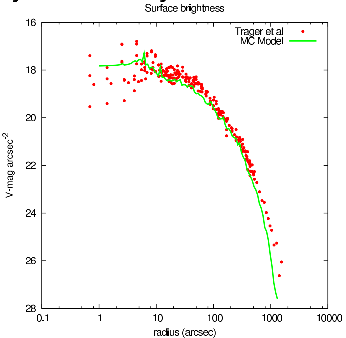
M4: Initial conditions

Monte Carlo simulation takes $\sim$ 1 day. Best initial conditions which we found in a multi-parameter search:

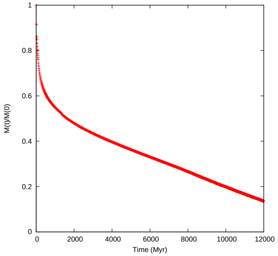
- $N \simeq 453000$
- $r_h \simeq 0.58$pc (half-mass radius)
- $f_b \simeq 0.07$ (binary fraction)

After 12 Gyr of evolution this gives fairly satisfactory fit to

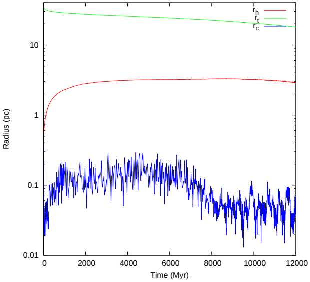
- Surface brightness profile $\rightarrow$
- Velocity dispersion profile
- Mass functions at two radii



Evolution of the Monte Carlo model of M4



(a) Mass (in units of the initial mass)

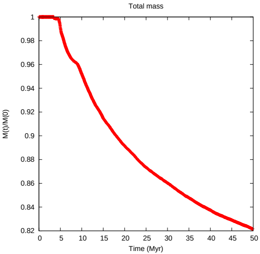


(b) Core, half-mass and tidal radii (*N*-body units)

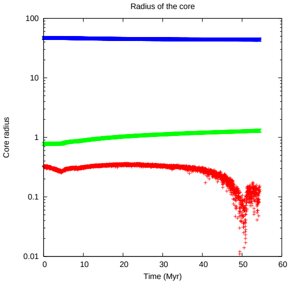
N-Body Models of M4

- A realisation of these initial conditions is at http://www.maths.ed.ac.uk/∼heggie/m4-fort.10.gz (31Mb).
- Initial unit of N-body time is 0.017 Myr.
- With NBODY6 each unit of time initially took about 3hrs, hence expect entire simulation would take 240 yr.
- Monte Carlo simulation shows that r_h increases from 0.58 pc to 3.2 pc, and N decreases from 453 000 to 80 900. Hence simulation will accelerate.
- If we assume wall-clock time W varies with time t as $dW \propto \frac{dt}{t_{rh}} N^{10/3}$ we find total duration of simulation ∼1–5 yr. However, most of the speedup is towards the end; after 2 years the simulation will have reached only 360 Myr. (Now, after 6 months, reached 56 Myr.)
- Other factors (e.g. increased proportion of unperturbed binaries) may accelerate the calculation further.

Evolution of the *N*-body model of M4

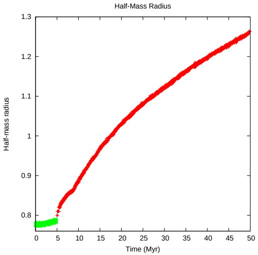


(c) Mass (in units of the initial mass)

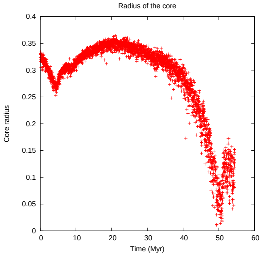


(d) Core, half-mass and tidal radii (*N*-body units)

Results 2. Half-mass radius, and radius of the core



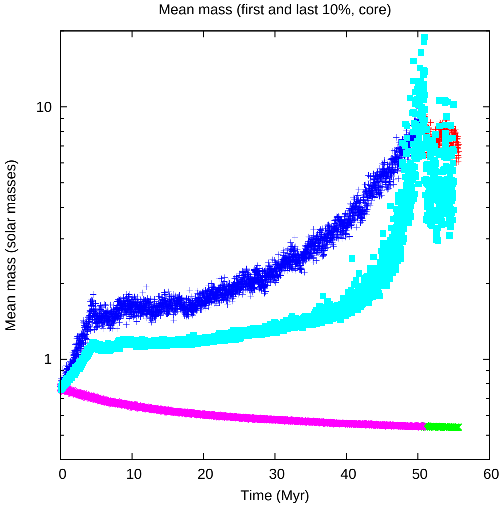
(e) Half-mass radius



(f) Core radius

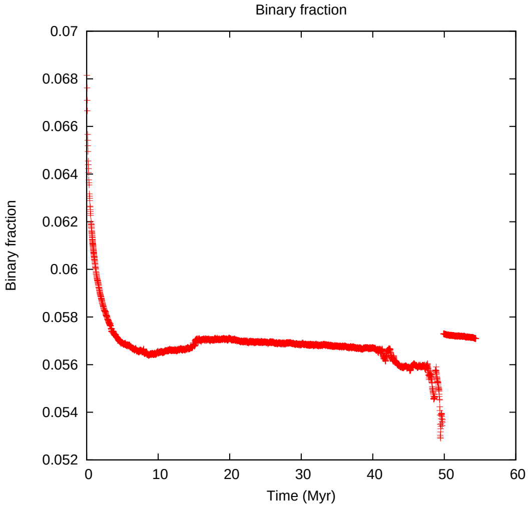
Increase caused by “heating” due to rapid onset of mass loss by stellar evolution at 4.5Myr, finally by formation and hardening of binaries in the core of stellar-mass black holes

Results 2. Mass segregation



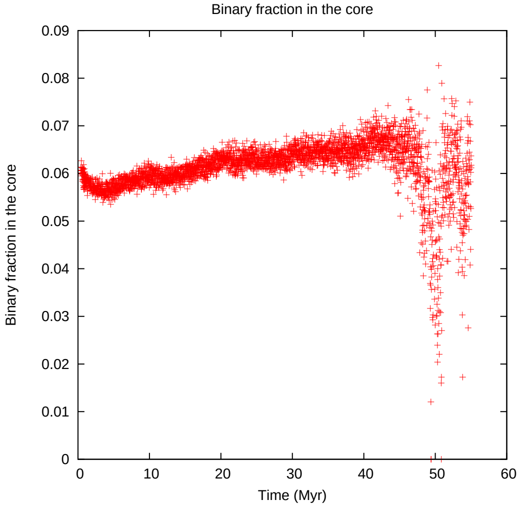
Note: initial relaxation time 60 Myr.

Results 3. Binary fraction



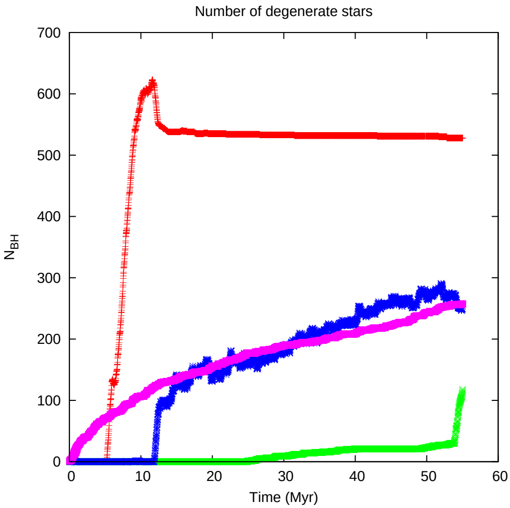
Note destruction of softest binaries (large semi-major axis)

Results 4. Binary fraction in the core

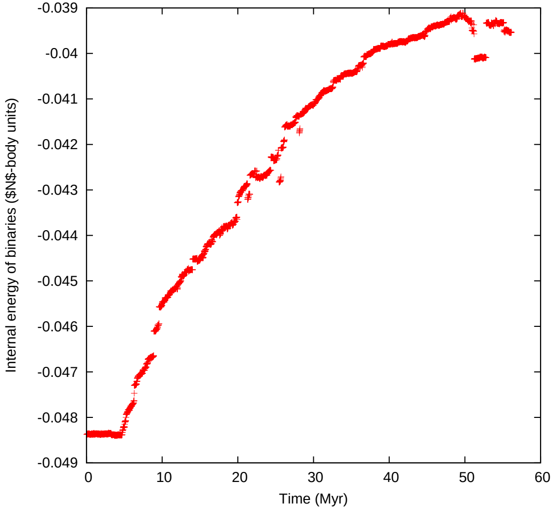


Note continued segregation of binaries into the core, eventually replaced by black holes

Results 5. Number of degenerate stars and collisions

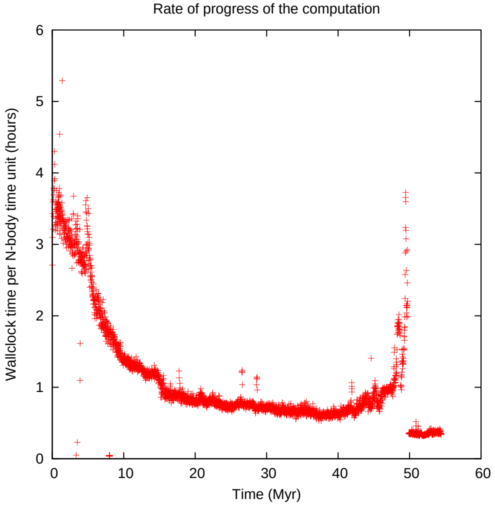


Results 6. Internal energy of binaries



Formation and ejection of black hole binary at 51Myr

Progress of the Simulation



Software, and entry-level hardware

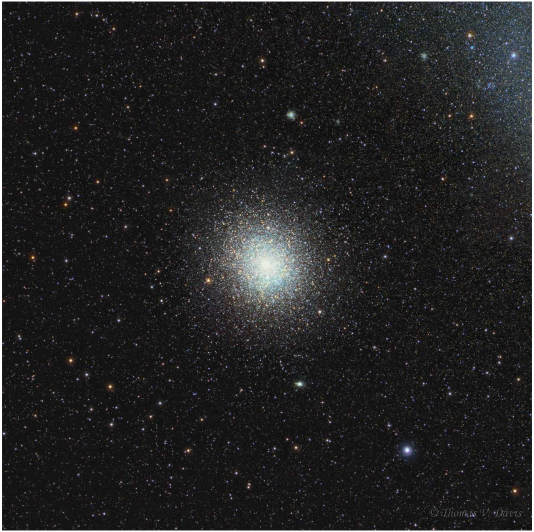
- NBODY6 (with OpenMP and GPU support): www.ast.cam.ac.uk/∼sverre/web/pages/nbody.htm
  -

(g) NVIDIA GeForce GTX 285 graphics card

(h) Super Micro 7046gt-trf server

The Future

- GPU-enabled parallel version of NBODY6++ (in progress)
- “Multiples” module in AMUSE (see S. McMillan’s Lecture 3)
- Clusters of GPUs, e.g. NAOC Beijing (image: P. Berczik)



...for what could be more beautiful than the heavens, which contain all beautiful things? (Copernicus, de Revolutionibus)

Summary of Lecture 3

- Two types of model: “snapshot” and evolutionary
- Fitting a King model
- Various observational constraints
- Challenge of computing an N-body simulation in a reasonable time
- Scaling N-body simulations
- Hénon’s Monte Carlo method
- Example: evolutionary simulation of M4
- Recent hardware developments