Accretion discs solutions in AGN

The principles of accretion. I

Shakura & Sunyaev (1973)

- Gas falling onto a compact object loses its potential energy, which is first converted into kinetic energy
- If the infall is not prevented, the gas will fall into the compact object without being able to radiate
- The gas has finite angular momentum: it cannot fall straight onto the compact object, since this is prevented by the momentum barrier
- Through **viscous friction** with other gas particles and by the resulting **momentum transfer** outward, the gas will assemble in a disc oriented perpendicular to the direction of the angular momentum vector
- The frictional forces in the gas are expected to be much smaller than the gravitational force, hence the disc will locally rotate with approximately the Keplerian velocity. Each gas element interacts with the surrounding elements, thus redistributing the energy and placing at the *minimum state of energy corresponding to the circular orbit*
- Since a Keplerian disc rotates differentially (i.e., angular velocity depends on radius), the gas in the disc will be heated by *internal friction*, which causes a slight deceleration of the rotational velocity, whereby the gas will slowly move inwards
- The energy source for heating the gas in the disc is provided by the inward motion, namely *the conversion of potential energy into kinetic energy*, which is then converted into *internal energy (heat) by friction* → emission of the disc as a blackbody in the case of optically thick disc

The principles of accretion. II

According to the virial theorem, $2T+U=0$ → half of the potential ('binding') energy released is converted into kinetic energy (the rotational energy of the disc), the other half can be converted into internal energy (heat) – see the following slides

Potential gravitational energy → kinetic energy of the infalling gas → heat (through viscosity among disc annuli) → thermal emission from the disc

Mass of the compact object (e.g., BH)

$$\Delta E = \frac{GM_\bullet m}{r} - \frac{GM_\bullet m}{r+\Delta r} \approx \frac{GM_\bullet m}{r}\frac{\Delta r}{r}$$

Energy released due to a mass m falling from a radius r+Δr to r

$M_\bullet$ is assumed to dominate the gravitational potential

Half is converted into heat: $E_{heat}=\Delta E/2$

$$E_{heat} = \frac{1}{2}\frac{GM_\bullet m}{r^2}\Delta r$$

$$\Delta L = \frac{GM_\bullet \dot{m}}{2r^2}\Delta r$$

Further details later in the lesson

The principles of accretion. III

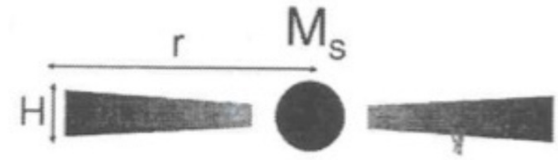
- As described in the *optically thick, geometrically thin accretion disc theory* (see following slides), the temperature profile of the resulting emission from the AD is independent on the detailed mechanism of dissipation (the equations do not explicitly contain the *viscosity term*)
- The emission of the AD increases inwards and, at first approximation, can be considered the *superposition of black bodies* consisting of rings with different radii at different temperatures. The resulting shape does not have a Planck shape but instead shows a much broader energy distribution
- The resulting spectrum from such an optically thick accretion disc is fairly flat, where the lower and upper boundaries of the frequency interval are determined by the lowest and highest temperatures (at the outer and inner radius) of the AD
- Most of the luminosity comes from the inner part of the disc (since T prop. $R^{-3/4}$) and thus depends on how far the disc extends inside (i.e., ISCO…)

*The physical mechanism that is responsible for the **viscosity** is unknown*. Molecular viscosity likely provides a minor contribution. Possibly, the viscosity is produced by turbulent flows in the disc and/or by magnetic fields (*MRI, magneto-rotational instability*), which become spun up by differential rotation and thus amplified, so that these fields may act as an effective **friction**

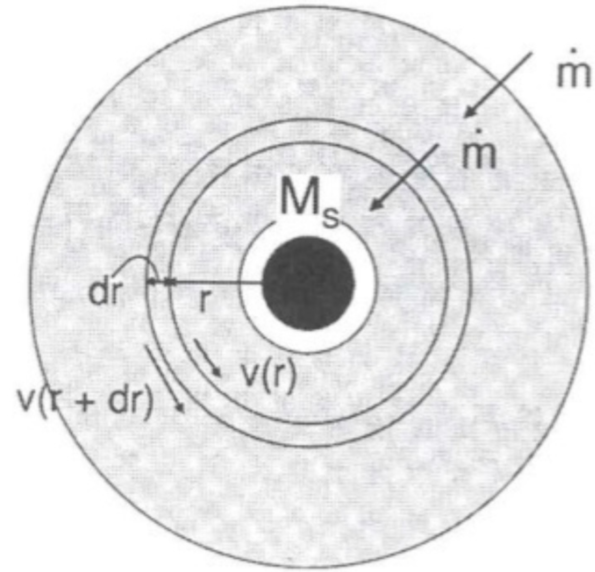
Geometrically thin discs. I

ASSUMPTIONS:

- $M_{disc} << M_{BH}$ (M_S here)
- $H << r$ (vertical size << size of the disc)



Keplerian rotation implies a velocity profile which is not solid rotation → there exist shear forces and, hence, transfer of angular momentum from annulus to annulus.
If the angular momentum is transferred outward, then the matter can fall onto the compact object and be accreted



We can see the accretion disc + compact object system as an *efficient machine for slowly lowering material in the gravitational potential of the accreting object and 'extracting' energy as radiation*. A vital part consists in the process converting the orbital kinetic energy into heat

Main adopted textbooks

- *Accretion Power in Astrophysics* (Frank, King, Raine)
- *High Energy Astrophysics* (Courvoisier)
- *High Energy Astrophysics* (Longair)
- *High Energy Astrophysics* (Melia)

Geometrically thin discs. II

Random motion of particles of velocity $\overline{v}$ and *free mean path* λ (where λ << size of the disc)

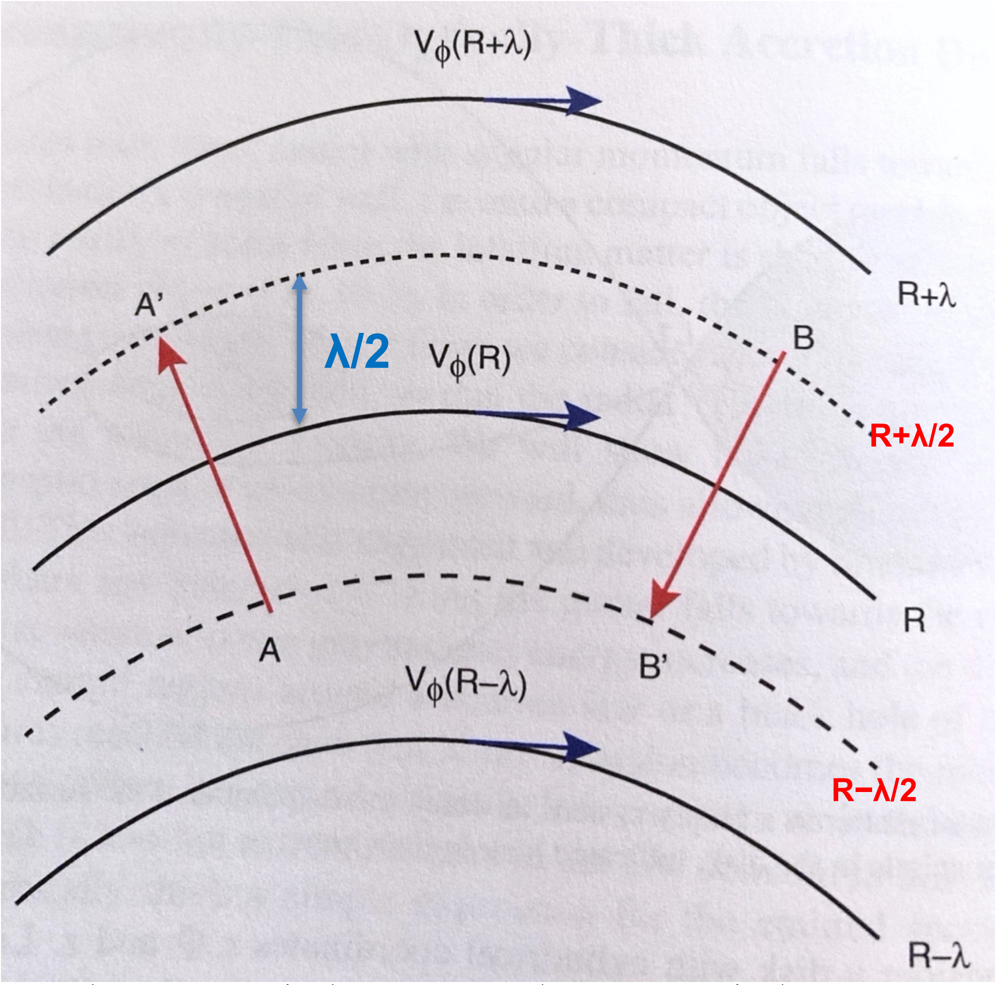
Particles A move from the inside annulus towards the outside annulus, and B move in the opposite direction

The orbital velocity v_ϕ at a given radius is linked to the angular velocity Ω(r)

$$v_\phi = \Omega(r) \times r$$

The velocity increases at smaller radii (i.e., the annuli closer to the compact object move faster than the outer ones)

Particle transport across a boundary

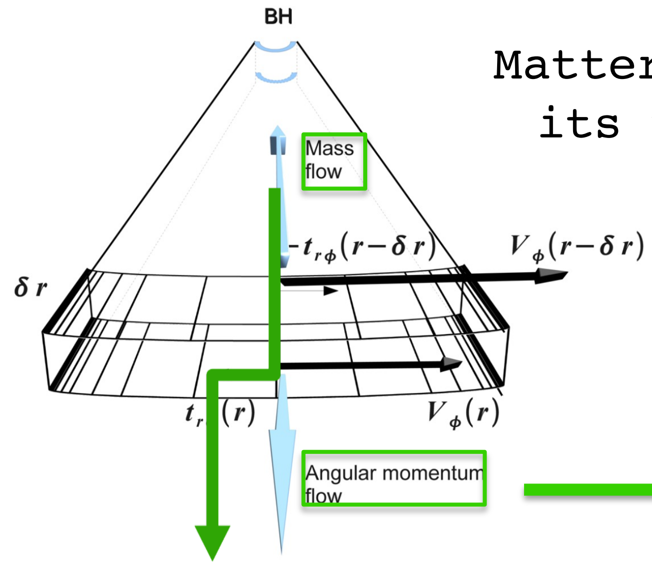


at the position r-λ/2

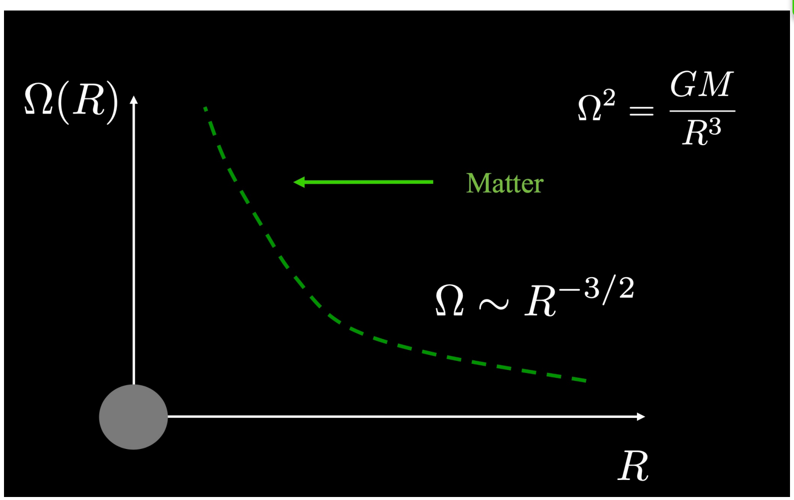
$$v_\phi\,(r-\lambda/2) = \Omega\,(r-\lambda/2) \times (r-\lambda/2) > v_\phi(r+\lambda/2)$$

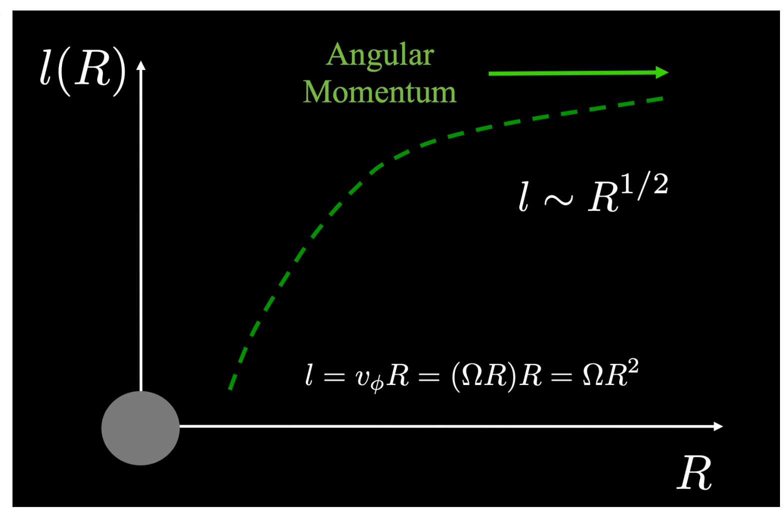
Summarizing

Armijo AD theory review



Matter is flowing to the center (increasing its velocity), while angular momentum is transported outwards





Geometrically thin discs. III

Particles A transport outwards the outside ring a *linear momentum* $M_A \times v_\Phi$ (r-λ/2), larger than the momentum transported in the opposite direction by the particles B, M_B v_Φ (r+λ/2).

The **net transfer of angular momentum L=m v_Φ r** across r per unit time due to *viscous transport* (due to the different rotational velocity of adjacent annuli) angular is given by

The change of torque across a ring = change of angular momentum of the ring

G(r)=torque

$$\frac{\Delta L}{\Delta t} = G(r) = \dot{M}_A \times (r+\lambda/2) \times v_\phi(r-\lambda/2) - \dot{M}_B \times (r-\lambda/2) \times v_\phi(r+\lambda/2) =$$

$$= \dot{M}_A \times (r+\lambda/2) \times \Omega(r-\lambda/2) \times (r-\lambda/2) - \dot{M}_B \times (r-\lambda/2) \times \Omega(r+\lambda/2) \times (r+\lambda/2) =$$

$$= \dot{M}_A \times (r^2-\lambda^2/4) \times \Omega(r-\lambda/2) - \dot{M}_B \times (r^2-\lambda^2/4) \times \Omega(r+\lambda/2) \sim$$

$$\sim \dot{M}_A \times r^2 \times \Omega(r-\lambda/2) - \dot{M}_B \times r^2 \times \Omega(r+\lambda/2)$$

Assuming the randomness of the process and the absence of net transport of matter due to the $\overline{v}$ random movements (i.e., matter crosses the surface R=constant at equal rate in both directions, the flow is in equilibrium) and defining **Σ=ρ H = surface mass density of the disc** [g/cm^2]

$$\dot{M}_A = \dot{M}_B = (2\pi r H)\rho(r)\overline{v} = 2\pi r \Sigma \overline{v}$$

Geometrically thin discs. IV

$$\nu = \lambda \overline{v}$$

kinematic viscosity [cm^2/s]

mean free path

thermal speed of the molecules

case of molecular transport in shearing motion

$$\nu = \lambda \overline{v}$$

spatial scale or characteristic wavelength of the turbulence

typical velocity of the eddies

case of turbulence

Geometrically thin discs. V

$$G(r) = 2\pi r \Sigma \bar{v} r^2 [\Omega(r - \lambda/2) - \Omega(r + \lambda/2)] =$$

$$= 2\pi r \Sigma \bar{v} r^2 \lambda \frac{d\Omega}{dr} =$$

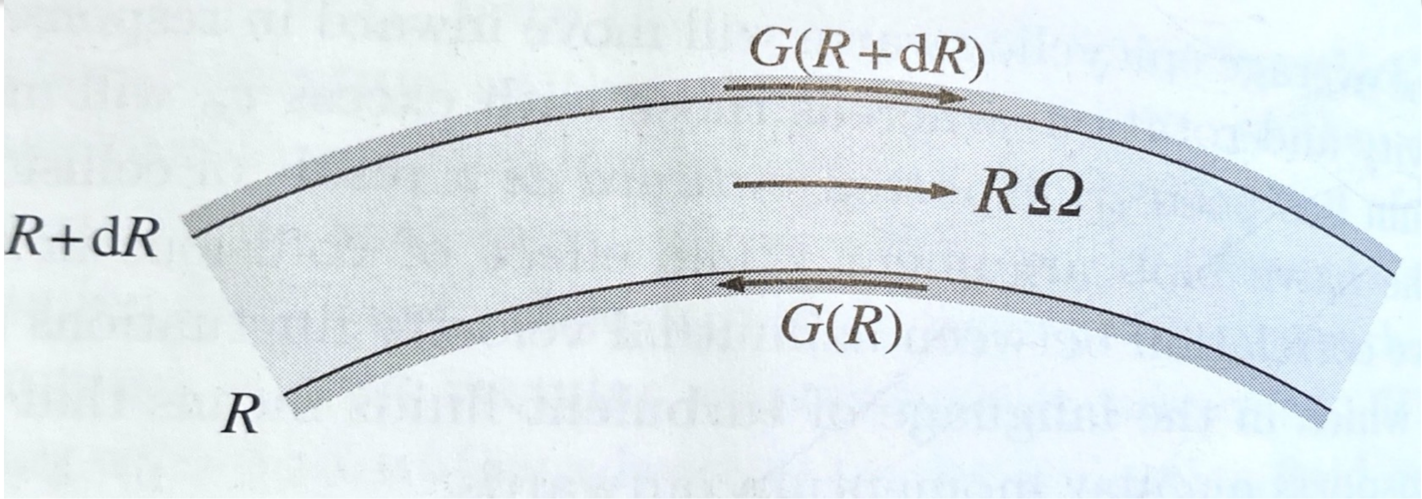
$$= 2\pi \Sigma \nu r^3 \frac{d\Omega}{dr}$$

$$G(r) = 2\pi r \Sigma \nu r^2 \frac{d\Omega}{dr} = 2\pi r \Sigma \nu r^2 \Omega'$$

Does it agree with our expectations?

- Rigid rotation: Ω' (=dΩ/dr)=0 → G=0 (the torque vanishes, no shearing of elements)
- If Ω(r) decreases outwards (as in Keplerian discs), G(r) is negative → the inner rings lose angular momentum to the outer ones and the gas slowly spirals in (towards the central object)

Geometrically thin discs. VI



Net torque on a ring of gas between R and R+dR. The ring is subject to competing torques since it has an inner and a outer radius

$$G(R+dR) - G(R) = \frac{\partial G}{\partial R} dR$$

$$\Omega \frac{\partial G}{\partial R} dR = \left[\frac{\partial (G\Omega)}{\partial R} - G\Omega^{'} \right] dR$$

since $\frac{\partial (G\Omega)}{\partial R} = G \frac{\partial \Omega}{\partial R} + \Omega \frac{\partial G}{\partial R} = G\Omega' + \Omega \frac{\partial G}{\partial R}$

Geometrically thin discs. VII

$$\frac{\partial(G\Omega)}{\partial R}dR$$

is the rate of 'convection' of rotational energy through the gas by the torques

↓ if integrated over the whole disc

$$[G\Omega]_{outer\ edge}^{inner\ edge}$$

$$-G\Omega^{'}dR$$

local rate of loss of mechanical energy by the gas
→ it goes into internal (heat) energy

The viscous torques cause **viscous dissipation** within the gas at the rate reported above per ring of width dR
→ This energy will be irradiated by the upper and lower faces of the disc

Geometrically thin discs. VIII

Rate of viscous dissipation per unit plane surface area

$$D(R) = \frac{G\Omega' dR}{4\pi R\ dR} = \frac{G\Omega'}{4\pi R} = \frac{(2\pi R\Sigma\nu R^2\Omega')\Omega'}{4\pi R} = \frac{1}{2}\Sigma\nu(R\Omega')^2$$

2 plane faces (2 × 2πRdR)

Angular velocity has the Keplerian form

$$\Omega = \left(\frac{GM}{R^3}\right)^{1/2} \Rightarrow\ \Omega' = (3/2)\ (GM)^{1/2}R^{-5/2}$$

$$D(R) = \frac{9}{8}\nu\Sigma\frac{GM}{R^3}$$

Geometrically thin discs. IX
Magnitude of viscosity

It can be shown that the shear viscosity gives a force density term in the Φ direction which is dimensionally identical to the bulk viscous force density

$$f_{visc,shear} \sim \rho\lambda\overline{v}\frac{\partial^2 v_\phi}{\partial R^2} \sim \rho\lambda\overline{v}\frac{v_\phi}{R^2}$$

Compare this term with the inertia terms in the Euler equation

$$\rho(\partial v/\partial t + (v \cdot \nabla)v)$$

v: velocity field (flow speed); ρv · ∇v=convection of momentum through the fluid by velocity gradients

$$R_e = \frac{inertia}{viscous} \sim \frac{v_\phi^2/R}{\lambda\overline{v}v_\phi/R^2} = \frac{Rv_\phi}{\lambda\overline{v}}$$

[laminar vs. turbulent flows 'discriminator']

$\lambda\overline{v}$ → kinematic viscosity

Reynolds number: measures the importance of viscosity

R_e<<1: viscous forces dominate the flow; R_e>>1: the viscosity associated with the given λ and $\overline{v}$ is dynamically not important → the fluid is in a turbulent regime where a new type of viscosity 'replaces' (i.e., is much higher than) the molecular one (but it is challenging to properly treat)

Geometrically thin discs. X
Shakura & Sunyaev 1973 solution

'Prescriptions':

- The typical size of the largest turbulent components ('eddies) cannot exceed the disk thickness H ($\lambda_{turb} \lesssim H$)
- It is unlikely that v_{turb} is supersonic (i.e., higher than the *sound speed* C_S in that medium)

$$\nu = \alpha c_S H$$

α is expected to be ≲1 (α-prescription), maybe not constant all over the disc – useful 'parameterization' of our very limited knowledge of the disc and its properties and processes

→ semi-empirical approach to the viscosity problem

Geometrically thin discs. XI
Radial disc structure

$$\Omega = \Omega_K(R) = \left(\frac{GM}{R^3}\right)^{1/2}$$ Keplerian angular velocity

$$v_\phi = R\Omega_K(R)$$ Circular velocity

The gas is supposed to have also a small radial 'drift' velocity v_R, which is negative close to the central object (hence matter can be accreted).
In general, $v_R = v_R(R,t)$

The disc is characterized by a surface density $\Sigma(R,t)$, given by the integration of the density ϱ along the z direction (perpendicular to the plane of the disc)

→ *Conservation equations* for the mass and angular momentum transport in the disc due to radial drift motions

Geometrically thin discs. XII
Radial disc structure

Between radii R and R+ΔR

$2\pi R \Delta R \Sigma$ Total mass

$2\pi R \Delta R \Sigma R^2 \Omega$ Total angular momentum = mvR

mass $\quad$ v × R=(ΩR) × R

Rate of change of both quantities ←→ net flow from neighbouring annuli (R and R+ΔR)

Geometrically thin discs. XIII
Radial disc structure

Mass conservation (R, R+ΔR 'boundaries')

Σ at R+ΔR ↑ — v_R at R+ΔR ↑

$$\frac{\partial}{\partial t}(2\pi R\Delta R\Sigma) = 2\pi R\Sigma(R,t)v_R(R,t) - 2\pi(R+\Delta R)\Sigma(R+\Delta R,t)v_R(R+\Delta R,t) \sim$$

mass rate at R — mass rate at R+ΔR

$$\sim -2\pi\Delta R\frac{\partial}{\partial R}(R\Sigma v_R)$$

variations of (RΣvR) between R and R+ΔR

$$\frac{\Delta R}{\Delta t} = v_R$$

At a given ('fixed') radius (annulus), R does not show time variations

$$\Delta R \to 0 \Rightarrow R\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma v_R) = 0 \qquad (1)$$

$\frac{\partial}{\partial t}$ indicates temporal variations of the mass due to the net flow of mass through two neighbouring annuli (representing the 'boundaries' of the present calculation)

Geometrically thin discs. XIV
Radial disc structure

Angular momentum conservation
(including the transport due to net effects of the viscous torque G(R,t))

$$G(R+dR)-G(R)=\frac{\partial G}{\partial R}dR$$

in the annuli R — in the annuli R+ΔR

$$\frac{\partial}{\partial t}(2\pi R\Delta R\Sigma R^2\Omega)=2\pi R\Sigma(R,t)v_R(R,t)R^2\Omega(R)-2\pi(R+\Delta R)\Sigma(R+\Delta R,t)v_R(R+\Delta R,t)\times$$

$$\times\ (R+\Delta R)^2\Omega(R+\Delta R)+\frac{\partial G}{\partial R}\Delta R\sim$$

$$\sim-2\pi\Delta R\frac{\partial}{\partial R}(R\Sigma v_R R^2\Omega)+\frac{\partial G}{\partial R}\Delta R$$

$$\Delta R\to 0\Rightarrow R\frac{\partial}{\partial t}(\Sigma R^2\Omega)+\frac{\partial}{\partial R}(R\Sigma v_R R^2\Omega)=\frac{1}{2\pi}\frac{\partial G}{\partial R} \quad (2)$$

$\frac{\partial}{\partial t}$ indicates temporal variations of the angular momentum due to the net flow of a.m. through two neighbouring annuli

Geometrically thin discs. XV
Radial disc structure

$$G(r) = 2\pi R \Sigma \nu R^2 \Omega' \quad (3)$$

Using (1), (2) and (3) and assuming $\partial\Omega/\partial t = 0$ (valid for orbits in a fixed gravitational potential)

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma v_R) = 0 \quad (1)$$

$$R\frac{\partial}{\partial t}(\Sigma R^2 \Omega) + \frac{\partial}{\partial R}(R\Sigma v_r R^2 \Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial R} \quad (2)$$

assumption ∂Ω/ ∂t=0

$$\Omega R^3 \frac{\partial \Sigma}{\partial t} + R^3 \Sigma \frac{\partial \Omega}{\partial t} + \Omega R^2 \frac{\partial}{\partial R}(R\Sigma v_R) + R\Sigma v_R \frac{\partial}{\partial R}(R^2\Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial R}$$

$$\Omega R^2 \frac{R\partial \Sigma}{\partial t} + \Omega R^2 \frac{\partial}{\partial R}(R\Sigma v_R) + R\Sigma v_R (R^2\Omega)' = \frac{1}{2\pi}\frac{\partial G}{\partial R}$$

$$\Omega R^2 \left(\frac{R\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma v_R)\right) + R\Sigma v_R (R^2\Omega)' = \frac{1}{2\pi}\frac{\partial G}{\partial R}$$

Based on (1)

$$\Rightarrow R\Sigma v_R (R^2\Omega)' = \frac{1}{2\pi}\frac{\partial G}{\partial R}$$

Geometrically thin discs. XVI
Radial disc structure

$$R\Sigma v_R (R^2\Omega)' = \frac{1}{2\pi}\frac{\partial G}{\partial R} \qquad (4)$$

Combining (1) and (4) to 'remove' v_R dependence

$$R\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial R}(R\Sigma v_R) = -\frac{\partial}{\partial R}\left[\frac{1}{2\pi(R^2\Omega)'}\frac{\partial G}{\partial R}\right]$$

$$\rightarrow \frac{\partial \Sigma}{\partial t} = -\frac{1}{R}\frac{\partial}{\partial R}\left[\frac{1}{2\pi(R^2\Omega)'}\frac{\partial G}{\partial R}\right]$$

$$\Omega' = \frac{3}{2}(GM)^{1/2}R^{-5/2}$$

$$\frac{1}{2\pi(R^2\Omega)'} = \frac{1}{2\pi\frac{\partial}{\partial R}(R^2\Omega)} = \frac{1}{2\pi\frac{\partial}{\partial R}(R^2(GM)^{1/2}/R^{3/2})} = \frac{1}{2\pi\frac{\partial}{\partial R}(R^{1/2}(GM)^{1/2})} = \frac{1}{\pi(R^{-1/2}(GM)^{1/2})} = \frac{R^{1/2}}{\pi(GM)^{1/2}}$$

(a)

$$\frac{\partial G}{\partial R} = \frac{\partial}{\partial R}(2\pi R\nu\Sigma R^2\Omega') = \frac{\partial}{\partial R}(2\pi R^3\nu\Sigma\frac{3}{2}R^{-5/2}(GM)^{1/2}) = \frac{\partial}{\partial R}(3\pi R^{1/2}\nu\Sigma(GM)^{1/2}) = 3\pi(GM)^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})$$

(b)

Geometrically thin discs. XVII
Radial disc structure

combining (a) and (b) →

$$\frac{1}{2\pi(R^2\Omega)'}\frac{\partial G}{\partial R} = \frac{R^{1/2}}{\pi(GM)^{1/2}}3\pi(GM)^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2}) =$$

$$= 3R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})$$

For Keplerian orbits and using G(R)

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\{R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\}$$

Basic equation governing the **time evolution of surface density in a Keplerial disc**.
ν (viscosity) can be a function of Σ, R and *t*

$$R\Sigma v_R(R^2\Omega)' = \frac{1}{2\pi}\frac{\partial G}{\partial R} \rightarrow v_R = \frac{1}{2\pi}\frac{\partial G}{\partial R}\frac{1}{R\Sigma(R^2\Omega)'} =$$

$$= \frac{1}{2\pi}3\pi(GM)^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\frac{1}{R\Sigma\frac{\partial}{\partial R}(R^{1/2}(GM)^{1/2})} =$$

$$= \frac{3}{2}(GM)^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\frac{1}{(1/2)(GM)^{1/2}R^{1/2}\Sigma} =$$

$$= \frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})$$

Radial velocity

$$v_R = -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})$$

Following G(R) and (4)

Geometrically thin discs. XVII
Radial disc structure and timescales

Viscous (or radial drift) timescale: timescale in which matter diffuses through the disc over a distance R under the effect of viscous torques.
Σ changes on t_{visc} timescale

$$t_{visc} \sim R^2/\nu$$

$$v_R = -\frac{3}{\Sigma R^{1/2}} \frac{\partial}{\partial R}(\nu \Sigma R^{1/2}) \sim \frac{\nu}{R}$$

$$\Rightarrow \quad t_{visc} \sim R/v_R$$

Dynamical timescale: corresponds to the orbital timescale

Keplerian velocity

$$t_{dyn} = R/v_K = R/(R\Omega_K) = (1/\Omega_K) = (R^3/GM)^{1/2}$$

Hydrostatic equilibrium application to the disc

Consider the z-(vertical) direction in the structure of the disc, where essentially there is no flow – Euler equation, with all velocity terms neglected

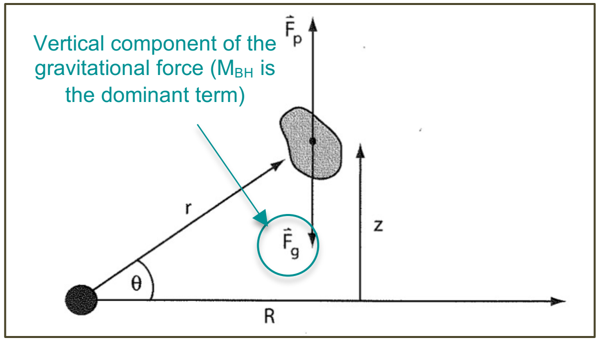
$$\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(\frac{GM}{(R^2+z^2)^{1/2}}\right) = -\frac{GMz}{(R^2+z^2)^{3/2}}$$

thin-disc approx.

$$z << R \rightarrow \frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{R^3}$$

If the typical scale height of the disc in the z direction is H

$$\frac{\partial P}{\partial z} \sim \frac{P}{H}, z \sim H \rightarrow \frac{P}{\rho} = c_S^2 = \frac{GMH^2}{R^3}$$



From F. Melia

$$c_S^2 = \frac{GM}{R}\left(\frac{H}{R}\right)^2 = v_K^2\left(\frac{H}{R}\right)^2$$

$$c_S = v_K\left(\frac{H}{R}\right) \rightarrow \frac{c_S}{v_K} = \left(\frac{H}{R}\right)$$

$$H << R \rightarrow \frac{c_S}{v_K} << 1$$

The *geometric thin disc approximation* requires that the local Keplerian velocity should be highly supersonic (maybe not satisfied throughout the whole disc)

Geometrically thin discs. XVIII
Radial disc structure and timescales

$$v_K = \frac{c_S R}{H} \rightarrow t_{dyn} = \frac{R}{v_K} = \frac{H}{c_S} \Rightarrow t_{visc} = \frac{R^2}{\alpha c_S H} = \frac{1}{\alpha}\left(\frac{R}{H}\right)^2 t_{dyn}$$

Thermal timescale: characteristic timescale needed for a system to recover thermal equilibrium

thermal energy content ($k\rho T/\mu m_H$) per unit surface area

$$t_{th} = \frac{\Sigma c_S^2}{D(R)} \sim \frac{\Sigma c_S^2}{\Sigma \nu G M / R^3} = \frac{c_S^2}{v_K^2}\frac{R^2}{\nu} = \left(\frac{H}{R}\right)^2 t_{visc}$$

viscous dissipation rate

$t_{dyn} \sim \alpha\, t_{th} \sim \alpha\, (H/R)^2\, t_{visc}$ → dynamical < thermal < viscous timescales

Geometrically thin discs. XIX
Steady thin disc

Changes in the radial structure of the disc occur on viscous timescale. In many systems, external conditions (e.g., mass transfer rate) change on $t>t_{visc}$. In this case, a stable disc will settle to a steady-state structure ($\partial/\partial t=0$ in the conservation of mass and angular momentum formulae, (1) and (2))

(1)

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma v_R) = 0 \rightarrow R\Sigma v_R = cons$$

This represents the constant inflow of mass through each point of the disc (because it is the integral of the mass conservation equation)

$$\dot{M} = 2\pi R\Sigma(-v_R)$$ **Mass accretion rate**

(2)

$$R\frac{\partial}{\partial t}(\Sigma R^2\Omega) + \frac{\partial}{\partial R}(R\Sigma v_r R^2\Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial R} \rightarrow R\Sigma v_r R^2\Omega = \frac{G}{2\pi} + \frac{C}{2\pi}$$

where C=constant

Geometrically thin discs. XX
Steady thin disc

$$G = 2\pi R\nu\Sigma R^2\Omega' \qquad \Omega' = d\Omega/dR$$

$$R\Sigma v_r R^2\Omega = \frac{2\pi R\nu\Sigma R^2\Omega'}{2\pi} + \frac{C}{2\pi}$$

$$-\nu\Sigma\Omega' = \Sigma(-v_r)\Omega + \frac{C}{2\pi R^3}$$

C is related to the rate at which the angular momentum flows into the compact object or to the couple exerted by the compact object on the inner edge of the disc

Consider the compact object being a star ($\star$) and the disc extending down to its surface. To prevent disruption, the star must rotate more slowly than the break-up speed at its equator

$$\Omega_\star < \Omega_K(R_\star)$$

The angular velocity of the disc material remains Keplerian and increases inwards, until it begins decreasing to $\Omega_\star$ in a sort of *boundary layer* of radial extent *b*

Geometrically thin discs. XXI
Steady thin disc

There exists a radius R=R$_\star$+b at which Ω'=dΩ /dR=0

$$\Omega(R_\star + b) = (\frac{GM}{R_\star^3})^{1/2}[1 + O(b/R_\star)]$$

Ω is very close to its Keplerian value

In practice, b<<R$_\star$ and thus Ω is very close to its Keplerian value at the point where Ω'=0

If b~R$_\star$, the thin-disc approximation breaks down at R=R$_\star$+b (we are not anymore in the condition of a 'boundary layer' but a sort of thick disc)

At R=R$_\star$+b

Ω at R$_\star$+b

$$0 = \Sigma(-v_r)\Omega + \frac{C}{2\pi R^3} \rightarrow C = 2\pi(R_\star + b)^3\Sigma v_r \Omega(R_\star + b) \sim$$

$$\sim 2\pi R_\star^3 \Sigma v_r \Omega(R_\star + b)$$

$$\dot{M} = 2\pi R\Sigma(-v_r)$$

$$\Rightarrow C = -\dot{M}(GMR_\star)^{1/2}$$

Geometrically thin discs. XXII
Steady thin disc

$$-\nu\Sigma\Omega' = \Sigma(-v_r)\Omega + \frac{C}{2\pi R^3}$$

$$\Omega = \Omega_K = (\frac{GM}{R^3})^{1/2}$$

$$\Omega' = -\frac{3}{2}R^{-5/2}$$

$$\rightarrow \nu\Sigma\Omega' = \Sigma v_r\Omega + \frac{\dot{M}(GMR_\star)^{1/2}}{2\pi R^3}$$

$$-\nu\Sigma\frac{3}{2}(GM)^{1/2}R^{-5/2} = \Sigma v_r(GM)^{1/2}R^{-3/2} + \frac{\dot{M}}{2\pi}(GM)^{1/2}R_\star^{1/2}R^{-3}$$

$$-\frac{3}{2}\nu\Sigma = \Sigma v_r R + \frac{\dot{M}}{2\pi}R_\star^{1/2}R^{-1/2}$$

$$\nu\Sigma = -\frac{2}{3}[\Sigma v_r R + \frac{\dot{M}}{2\pi}(\frac{R_\star}{R})^{1/2}]$$

$$\dot{M} = 2\pi R\Sigma(-v_R)$$

$$\nu\Sigma = \frac{\dot{M}}{3\pi}[1 - (\frac{R_\star}{R})^{1/2}]$$

Geometrically thin discs. XXIII
Steady thin disc

$$\nu\Sigma = \frac{\dot{M}}{3\pi}[1 - (\frac{R_\star}{R})^{1/2}]$$

$$D(R) = \frac{9}{8}\nu\Sigma\frac{GM}{R^3}$$

viscous dissipation per unit disc face area

$$D(R) = \frac{9}{8}\frac{\dot{M}}{3\pi}[1 - (\frac{R_\star}{R})^{1/2}]\frac{GM}{R^3} = \frac{3GM\dot{M}}{8\pi R^3}[1 - (\frac{R_\star}{R})^{1/2}]$$

The energy flux through the faces of a steady thin disc is independent on the ν (viscosity). Such independence relies on the fact that we adopted conservation laws to 'remove' the ν term, BUT *the other disc properties (Σ, v_r) do depend on ν*

$$D(R) = \frac{3GM\dot{M}}{8\pi R^3}[1 - (\frac{R_\star}{R})^{1/2}]$$

Geometrically thin discs. XXIV
Steady thin disc

Luminosity produced by the disc in the annulus (R_1, R_2)

2 faces of the disc

$$L(R_1, R_2) = 2\int_{R_1}^{R_2} D(R) 2\pi R dR = \frac{3GM\dot{M}}{2}\int_{R_1}^{R_2}[1-(\frac{R_\star}{R})^{1/2}]\frac{1}{R^2}dR$$

$$L(R_1, R_2) = \frac{3GM\dot{M}}{2}\{\frac{1}{R_1}[1-\frac{2}{3}(\frac{R_\star}{R_1})^{1/2}] - \frac{1}{R_2}[1-\frac{2}{3}(\frac{R_\star}{R_2})^{1/2}]\}$$

$$if \;\; R_1 = R_\star, R_2 \rightarrow \infty \Rightarrow L_{disc} = \frac{3GM\dot{M}}{2}\{\frac{1}{R_\star}\frac{1}{3} - 0\}$$

$$L_{disc} = \frac{GM\dot{M}}{2R_\star} = \frac{1}{2}L_{acc}$$

$$L_{disc} = \frac{GM\dot{M}}{2R_\star} = \frac{1}{2}L_{acc}$$

The matter dissipates half of its potential energy in falling from infinity to radius R
→ source of luminosity of the disc

Geometrically thin discs. XXV
Steady thin disc

$$D(R) = \frac{3GM\dot{M}}{8\pi R^3}[1 - (\frac{R_\star}{R})^{1/2}]$$

Another way to 'interpret' the dissipation rate

Total rate at which energy is dissipated in a ring of internal radius R and external radius R+dR

$$2 \times \;\; 2\pi R \, dR \, D(R) = \frac{3GM\dot{M}}{2R^2}[1 - (\frac{R_\star}{R})^{1/2}] \, dR$$

$$\frac{GM\dot{M} \, dR}{2R^2}$$

$$\frac{GM\dot{M}}{R^2}[1 - \frac{3}{2}(\frac{R_\star}{R})^{1/2}]dR$$

Rate of release of gravitational binding energy between R+dR and R

Net flow of energy into the annulus dR associated with the transport of angular momentum outwards (i.e., work done by the local torque on the exterior disc)

At any radius, the energy dissipation rate consists of the release of gravitational binding energy + energy loss due to angular momentum transport (outwards)

Geometrically thin discs. XXVI
Steady thin disc

For R>> $R_\star$, this viscous transport of energy liberated lower down in the potential well is twice as important as the local binding energy loss $\left[\frac{GM\dot{M}}{R^2} \; vs \; \frac{GM\dot{M}}{2R^2}\right]$

→ The viscous transport has a key role in redistributing the energy release within the disc, but the total rate cannot change (i.e., it is preserved)

'GENERAL' FORM

$$L\ dR = -(\frac{dE}{dt})\ dR = \frac{3GM\dot{M}}{2R^2}[1-\beta(\frac{R_{in}}{R})^{1/2}]\ dR$$

- $R_{in}=R_{ISCO}$
- β<1: depends on whether the accretion fow within the ISCO has significant magnetic content to connect it to the disc flow further out (Krolik 1999)

Geometrically thin discs. XXVII
Steady thin disc

$$L = \int_{R_{in}}^{\infty} \frac{3GM\dot{M}}{2R^2}[1 - \beta(\frac{R_{in}}{R})^{1/2}]\, dR =$$

$$= \frac{3}{2}GM\dot{M}[\frac{2\beta(\frac{R_{in}}{R})^{3/2} - 3(\frac{R_{in}}{R})}{3R_{in}}]_{R_{in}}^{\infty} =$$

$$= \frac{3}{2}GM\dot{M}(-\frac{2\beta}{3R_{in}} + \frac{1}{R_{in}})$$

$$\rightarrow \; L = (\frac{3}{2} - \beta)\frac{GM\dot{M}}{R_{in}}$$

$$L = (\frac{3}{2} - \beta)\ \frac{GM\dot{M}}{R_{in}}$$

Geometrically thin discs. XXVIII
Steady thin disc, radial velocity

We have seen that in case of thin-disc condition, the circular matter velocity v_Φ will be very close to the Keplerian velocity v_K

$$H << R \rightarrow \frac{c_S}{v_K} << 1$$

What about the radial velocity?

$$\dot{M} = 2\pi R\Sigma(-v_R)$$

$$\nu\Sigma = \frac{\dot{M}}{3\pi}[1 - (\frac{R_\star}{R})^{1/2}]$$

$$v_R = -\frac{\dot{M}}{2\pi R\Sigma} = -\frac{\dot{M}}{2\pi R\frac{\dot{M}}{3\pi\nu}[1 - (\frac{R_\star}{R})^{1/2}]} = -\frac{3\nu}{2R}[1 - (\frac{R_\star}{R})^{1/2}]^{-1}$$

$$v_R = -\frac{3\nu}{2R}[1 - (\frac{R_\star}{R})^{1/2}]^{-1}$$

Geometrically thin discs. XXIX
Steady thin disc, radial velocity

$$v_R \sim \frac{\nu}{R} = \alpha \; c_S \; \frac{H}{R} << c_S$$

$$\nu = \alpha c_S H$$ SS73 ‘prescription’

The shear viscosity leads to an inexorable transfer of angular momentum outwards and a diffusion of mass inwards
The dominant velocity at any given radius is always the orbital one

$\mathcal{M}$=Mach number = v_Φ/c_S

$$v_\phi \sim (\frac{GM}{R})^{1/2}$$ = v_K

$$\frac{c_S}{v_K} = \frac{H}{R} \rightarrow H \sim \frac{c_S}{v_\phi} \; R = \frac{R}{\mathcal{M}}$$

$$v_R \sim \frac{\alpha \; c_S}{\mathcal{M}}$$

Geometrically thin discs. XXX
The emitted spectrum

If the disc is optically thick, each element of the disc face radiates roughly as a blackbody with temperature T(R)

$$\sigma\, T^4(R) = D(R) = \frac{3GM\dot{M}}{8\pi R^3}[1 - (\frac{R_\star}{R})^{1/2}]$$

$$\rightarrow T(R) = \{\frac{3GM\dot{M}}{8\pi R^3 \sigma}[1 - (\frac{R_\star}{R})^{1/2}]\}^{1/4}$$

$$R >> R_\star \rightarrow T = (\frac{3GM\dot{M}}{8\pi R_\star^3 \sigma})^{1/4}(\frac{R}{R_\star})^{-3/4}$$

More energy is dissipated close to the BH than far from it, i.e., more energy per unit area → regions close to the BH are hotter (T_{BB} increases at small radii)

commonly used formula $T(R) = \{\frac{3GM\dot{M}}{8\pi R^3 \sigma}[1 - (\frac{R_{in}}{R})^{1/2}]\}^{1/4} \propto R^{-3/4}$

Geometrically thin discs. XXXI
The emitted spectrum

$$I_\nu = B_\nu[T(R)] = \frac{2h\nu^3}{c^2} \; \frac{1}{e^{h\nu/kT(R)} - 1}$$

[erg/cm^2/s/Hz/sr]

The atmosphere of the disc (i.e., the part of the disc at optical depths $\tau \leqslant 1$ to infinity) is neglected in the redistribution of the radiation over frequency

D: distance of the observer
i: angle between the line-of-sight and the normal to the disc plane
A ring of radii R, R+dR subtends a solid angle $2\pi\, R dR \cos(i)/D^2$

$$F_\nu = \frac{2\pi \cos i}{D^2} \int_{R_\star}^{R_{out}} I_\nu \; R \, dR = \frac{4\pi h \cos i \; \nu^3}{c^2 D^2} \int_{R_\star}^{R_{out}} \frac{R \, dR}{e^{h\nu/kT(R)} - 1}$$

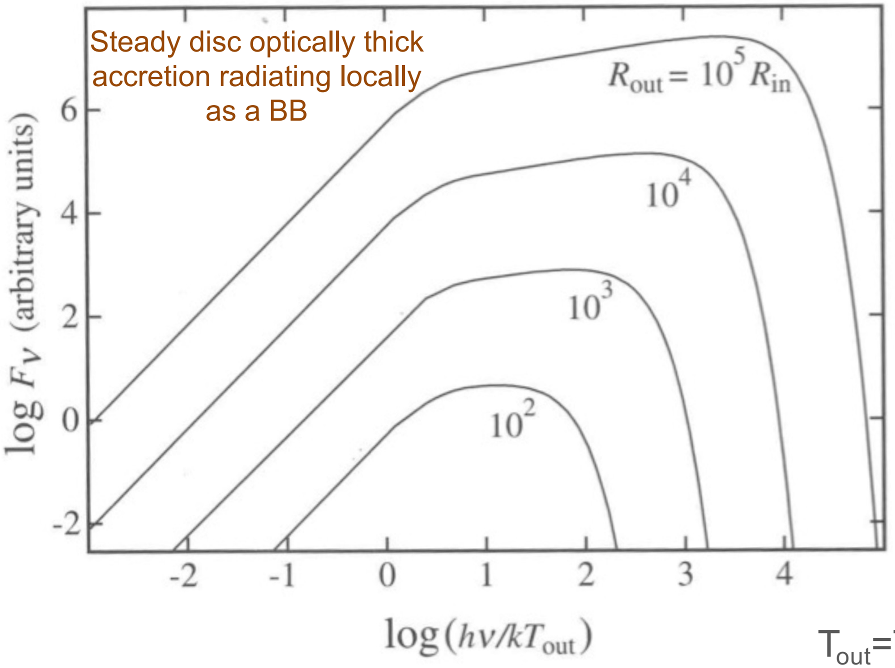
No dependence on viscosity

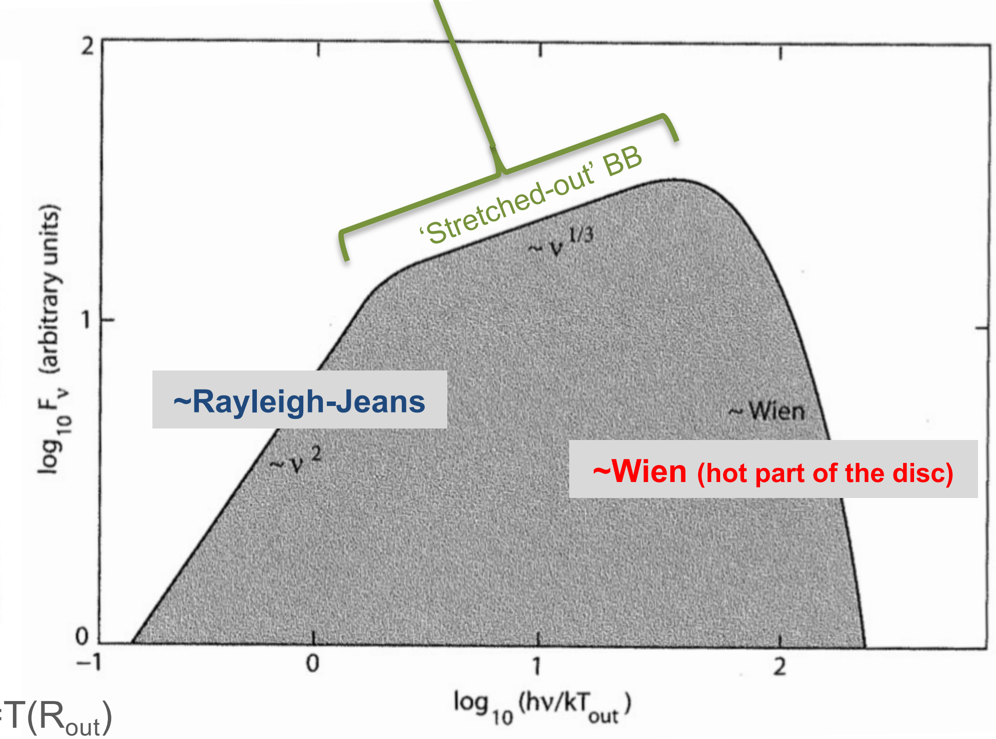
Geometrically thin discs. XXXII
The emitted spectrum

This is a valid solution for all sources in accretion having an optically thick accretion disc. For non-magnetic white dwarf and neutron stars, $R_{in}=R_\star$; for a magnetic white dwarf, it can be demonstrated that R_{in}=magnetospheric radius. For BHs, $R_{in}=R_{min}$ (ISCO, …)

At intermediate frequencies: the upper boundary of the integral is $h\nu/kT(R_{out})>>1$, while the lower boundary is $h\nu/kT(R_\star)<<1$

$$x = \frac{h\nu}{kT(R)} \sim \frac{h\nu}{kT_\star}\left(\frac{R}{R_\star}\right)^{3/4} \rightarrow F_\nu \propto \nu^{1/3}\int_0^\infty \frac{x^{5/3}}{e^x-1}dx \propto \nu^{1/3}$$

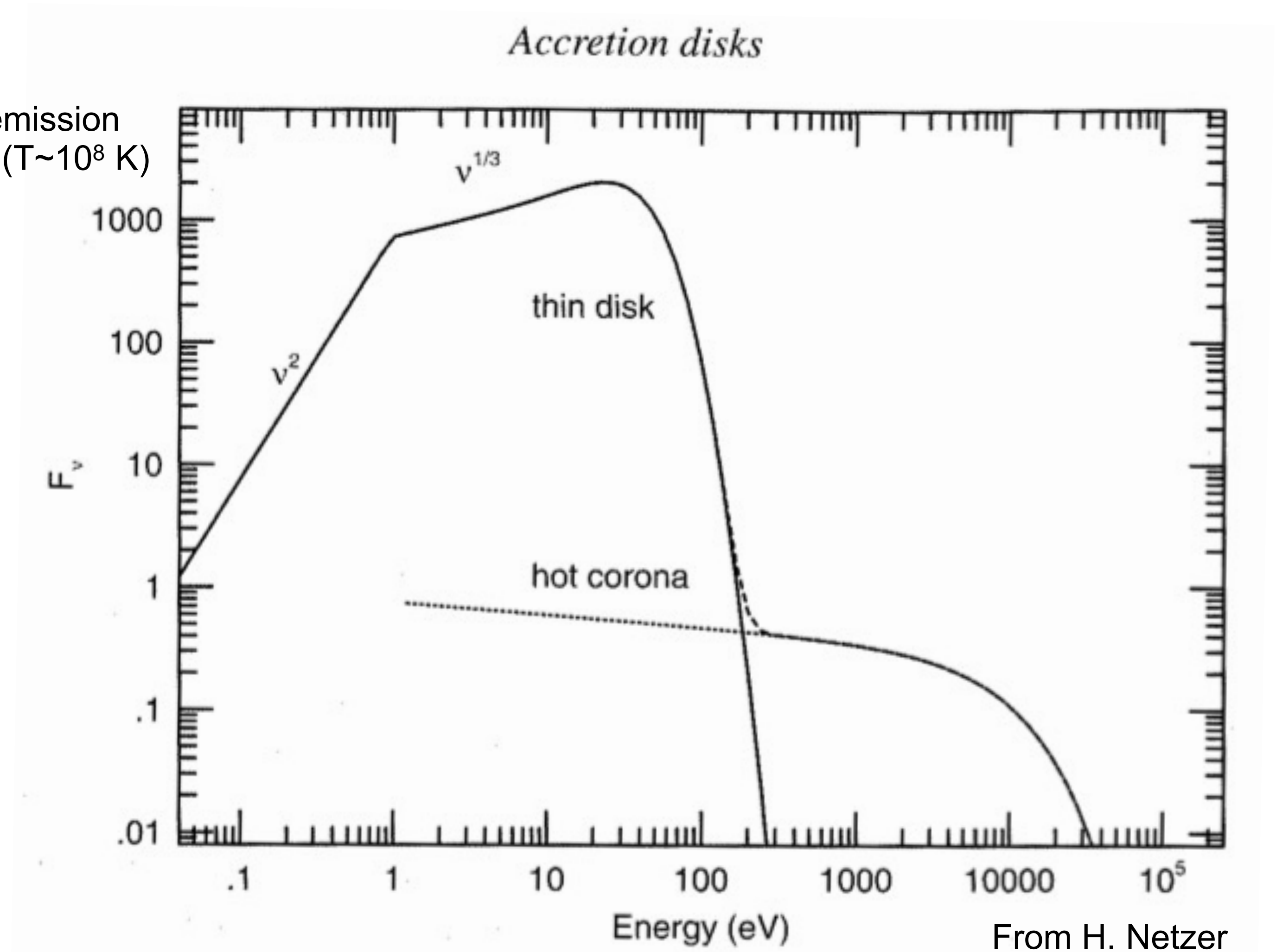




$T_{out}=T(R_{out})$

Geometrically thin discs. XXXIII
The emitted spectrum

Including the X-ray emission due to the hot corona ($T \sim 10^8$ K)



From H. Netzer

Geometrically thin discs. XXXIV
The outer boundary of the disc

What about R_{out}? – see the lesson on torus and obscured AGN

R_{OUT}~*self-gravity radius* which is the location where the local gravity exceeds the vertical component of the central BH gravity and the disc becomes unstable

$$R_{SG} \sim \; 1680 \; M_9^{-\frac{2}{9}} \alpha^{\frac{2}{9}} [\frac{L_{AGN}}{L_{Edd}}]^{\frac{4}{9}} [\frac{\eta}{0.1}]^{-\frac{4}{9}} \; R_g$$

$M_9 = M_{BH}/10^9\ M_\odot$
α=viscosity parameter
η=radiation efficiency

→ R_{SG}~0.04pc in case of L_{AGN}/L_{Edd}~0.1, $10^9\ M_\odot$ BH

The disc starts fragmentation into clouds moving in the same general plane beyond R_{SG}

Geometrically thin discs. XXXV
The local structure of thin discs

In the thin-disc approximation, at a given radius both the pressure and the temperature gradients are essentially vertical (the radial and vertical structures are largely decoupled).

Hydrostatic equilibrium $\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{R^3}$ provides

$$\rho(R,z) = \rho_c(R)\ e^{-z^2/2H^2}$$

density of the central disc plane z=0 (z=vertical direction)

$$\rho = \Sigma/H$$

$$H = R\ c_S/v_\phi$$

$$c_S^2 = P/\rho$$

T_C (central T) relies on an energy equation relating the energy flux in the vertical direction (e.g., radiative transport) to the rate of generation of energy by viscous dissipation

Stefan-Boltzmann constant

$$P = \frac{k\ \rho\ T_C}{\mu m_P} + \frac{4\sigma}{3c}T_C^4$$

Assumption: $T(R,z) \sim T_C(R) = T(R,0)$

Gas pressure

Radiation pressure

Geometrically thin discs. XXXVI
The local structure of thin discs

Consider radiative transport + disc medium assumed as 'plane-parallel' at each radius (so that the temperature gradient is effectively in the z-direction).
The flux of radiant energy at the surface z=constant is

$$F(z) = \frac{-16\ \sigma\ T^3}{3\ k_R\ \rho}\frac{\partial T}{\partial z}$$

Rosseland mean opacity

optical depth

$$\tau = \rho H k_R(\rho, T_C) = \Sigma k_R >> 1$$

It is implicitly assumed that the medium is optically thick

Energy balance: radiated flux = volume rate of energy production by viscous dissipation Q^+

$$\frac{\partial F}{\partial z} = Q^+ \rightarrow F(H) - F(0) = \int_0^H Q^+(z)dz = D(R)$$

surface temperature

$$F(z) \sim\ (4\sigma/3\tau)\ T^4(z)$$

$$If\ T_C^4 >> T^4(H) \rightarrow\ (4\sigma/3\tau)\ T_C^4 = D(R)$$

Geometrically thin discs. XXXVII

The local structure of thin discs – summarizing equations

$$[1]\ \rho = \frac{\Sigma}{H}$$

$$[2]\ H = \frac{c_S R^{3/2}}{(GM)^{(1/2)}}$$

$$[3]\ c_S^2 = \frac{P}{\rho}$$

$$[4]\ P = \frac{k\rho T_C}{\mu m_P} + \frac{4\sigma}{3c} T_C^4$$

$$[5]\ \frac{4\sigma T_C^4}{3\tau} = \frac{3GM\dot{M}}{8\pi R^3}[1 - (\frac{R_\star}{R})^{1/2}]$$

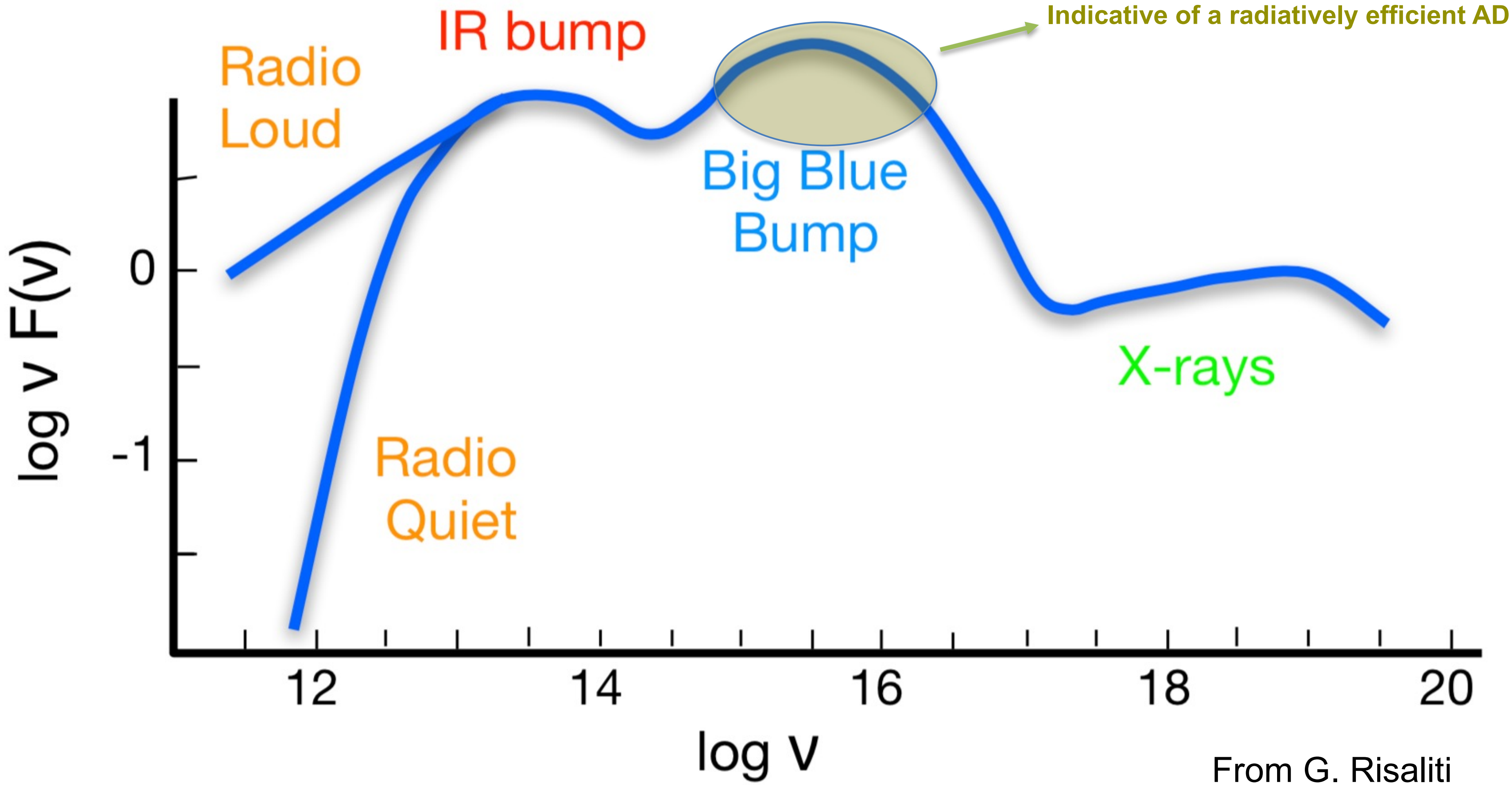
$$[6]\ \tau = \Sigma k_R(\rho, T_C) = \tau(\Sigma, \rho, T_C)$$

$$[7]\ \nu\Sigma = \frac{\dot{M}}{3\pi}[1 - (\frac{R_\star}{R})^{1/2}]$$

$$[8]\ \nu = \nu(\rho, T_C, \Sigma, \alpha, ...)$$

Solving these equations provide the derivation of the radial drift velocity

AGN
Spectral Energy Distribution (SED)



From G. Risaliti

What about the other ‘flavours’ of discs?

Radiation inefficient accretion flows: Advection-Dominated Accretion Flows (ADAF). I

So far we have discussed thin-disc solutions, where all of the gravitational binding energy is radiated away

There are cases where the radiation efficiency of the accretion flow is insufficient to emit all the E_{grav} liberated in the disc, i.e., the radiation transfer rate in the accreted matter is insufficient to carry all the energy to the surface of the flow where it can be radiated → *part of the energy is then advected with the flow* (which is therefore *adiabatic*), instead of being radiated from, and the accretion flow ultimately is accreted beyond the horizon (in BHs).

The flow retains most of its energy and does not radiate/cool efficiently

→ **ADAF solutions** (inefficient-disc solutions → lower luminosities than geometrically thin, optically thick Shakura-Sunyaev discs) – also called **RIAF** (Radiatively Inefficient Accretion Flows).

Often mentioned (with some differences) as ADIOS (ADiabatic Inflow-Outflow Solutions), CDAF (Convection-Dominated Accretion Flow), …

Treatment as in *High Energy Astrophysics* (Courvoisier) and *The Physics and Evolution of Active Galactic Nuclei* (Netzer)

ADAF solutions. II

Accretion discs with very low accretion rates may have much lower densities than in SS73 discs. Since cooling is generally prop. to N_e^2, cooling is much less efficient, and the particles retain the dissipated gravitational energy for a longer time → particle temperatures can rise to the virial temperature

$$\frac{1}{2}mv^2 = 3kT$$

$$mv^2 = \frac{GMm}{R}$$

$$\rightarrow T = \frac{mv^2}{6k} = \frac{GMm}{6kR}$$

$$r = \frac{R}{R_S} = \frac{R}{2GM/c^2}$$

$$\rightarrow T = \frac{GMm}{6krR_S} = \frac{GMm}{6kr2GM/c^2} = \frac{mc^2}{12kr}$$

→

$$T_i = \frac{m_p c^2}{12kr} \sim \frac{10^{12}}{r} K$$ Ions

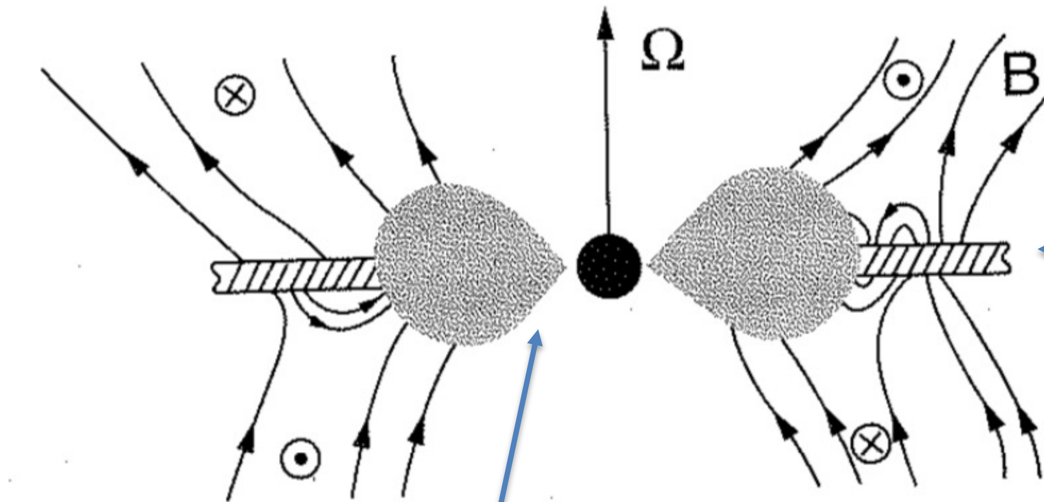
$$T_e = \frac{m_e c^2}{12kr} \sim \frac{10^9}{r} K$$ electrons

Coulomb collisions between ions and electrons in low-density gas are slower and less efficient in sharing the total kinetic energy

In ADAF, most of the E_{grav} locally deposited in the disc by viscous processes is stored as *internal energy of the ions* (because of their larger mass and because the electrons are more efficient coolants) → $T_{ions} \gg T_{electrons}$

The gas temperature close to the BH may reach very high values (T_{ions} if density is low and Coulomb interaction is inefficient)

ADAF solutions. III



Magnetic fields can be present: in such cases, $E_{pot}=E_{kin}+E_{magn}$

From Netzer

Low density, inefficient cooling in the innermost region. The gas does not cool and reach very high T → increase in the thickness of the disc (**the disc "inflates"**)

Inefficient conversion of gravitational to radiated electromagnetic energy because most of this is advected ('lost' in the BH) → low resulting luminosities

The accretion rate may not necessarily be very low: they can be extended and be characterized by a standard accretion disc in the outer regions (thus a dramatic change in the disc structure should be present) → two-temperature disc

ADAF solutions. IV

Radial flux of particles through the disc

Number of particles passing through a disc element per unit time $nvd\sigma$

Surface number density

Radial velocity v(r)

Area of the disc element

Change of internal energy per ion (in absence of mechanical work, $du = T\ ds$
Thermodynamics' first law)

Entropy

$$\frac{du_{adv}}{dr} = T\ \frac{ds}{dr} \rightarrow \frac{dU_{adv}}{dt} = nvT\frac{ds}{dr} = Q^{+} - Q^{-}$$

Total internal energy transfer across r per unit time and per unit surface

E_{grav} transferred to the plasma by viscous processes (D(R))

Energy radiated by the plasma by all processes

ADAF solutions. V

$$Q^+ = D(R) = \frac{3GM\dot{M}}{8\pi R^3}[1 - (\frac{R_\star}{R})^{1/2}]$$

for a Keplerian disc

Locally dissipated energy per unit surface

In case of SS73 (radiative efficient) discs, $Q^+ = Q^-$ (hence no net increase of internal energy locally)

In ADAF, $Q^+ >> Q^-$, and the mean T_{plasma} increases inward faster than in the previous (SS73) case. Almost all viscous energy is stored in the gas and deposited into the BH, i.e., an element of the gas is unable to radiate its thermal energy in less time than it takes to be transported through the disc onto the BH.
Cooling<<heating. For a given accretion rate, L(ADAF)<<L(cooling-dominated flow)

These are **Optically thin (2-T) ADAF** solutions with a very low, sub-Eddington accretion rate

There are also **Optically thick ADAF** solutions: objects are accreting at super-Eddington rate but radiate less than the standard disc (most of the radiation is trapped inside the disc and is carried toward the central object) – **slim discs**

ADAF solutions. VI

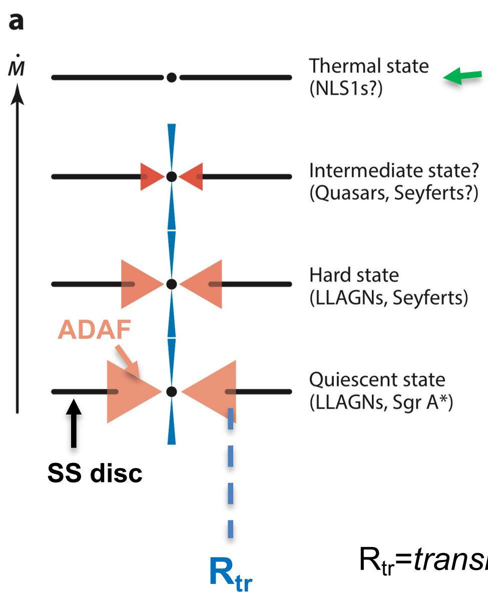
$$\dot{M} > \dot{M}_{crit}$$

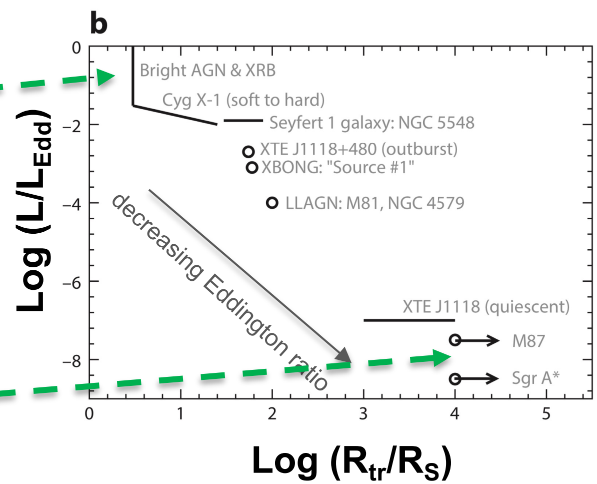
The system is well described by a standard efficient disc

$$\dot{M} < \dot{M}_{crit}$$

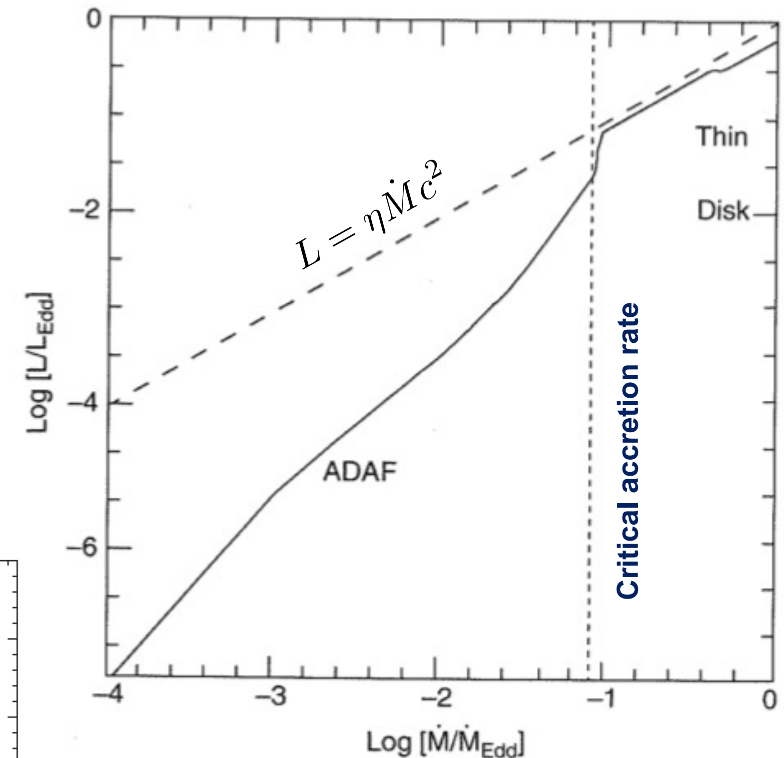
The system is in the ADAF regime (low-luminosity AGN, some LINERs, binaries in particular states, …)

$$\sim 0.4\alpha^2 \dot{M}_{Edd}$$





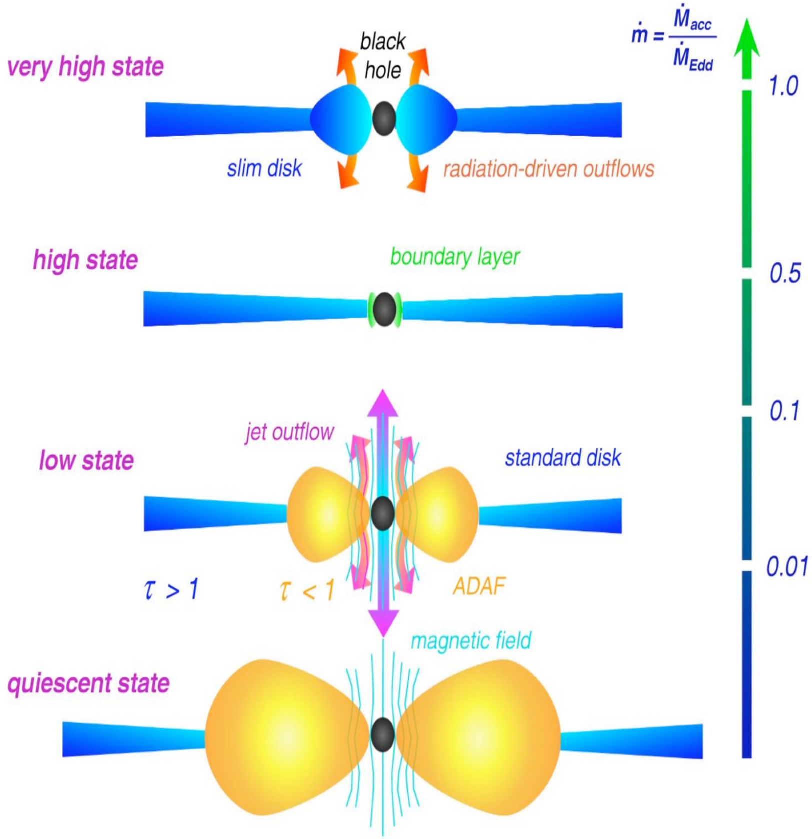
R_{tr}=*transition radius* between the hot flow and the thin disc

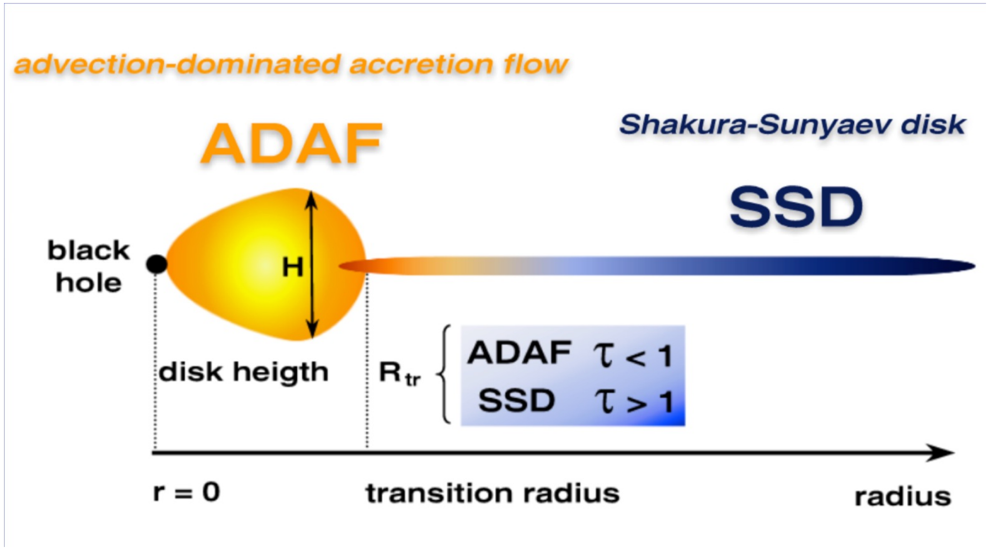


Narayan et al. (1998)

For details, see Yuan & Narayan et al. (2014, ARA&A)

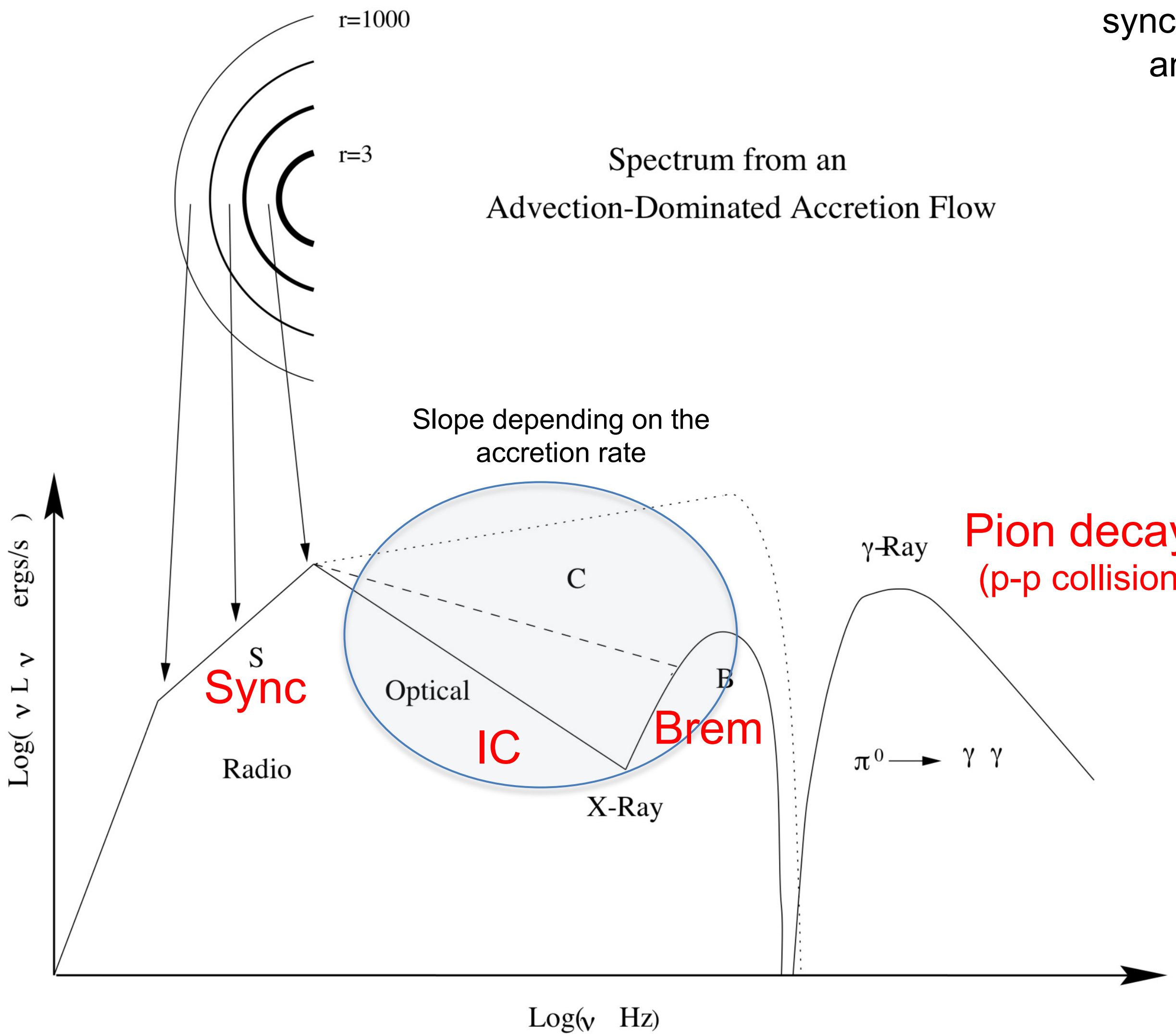
ADAF solutions. VII





ADAF solutions. VIII

Bremsstrahlung, synchrotron, Compton and pion-related processes



Narayan et al. (1998)